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The Propagation of a Sound Pulse in the Presence of a Semi-Infinite Open-Ended Channel. I

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THE PROPAGATION OF A SOUND PULSE IN THE PRESENCE OF A SEMI-INFINITE OPEN-ENDED CHANNEL. I

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CONTENTS

The behaviour of a sound pulse approaching and progressing beyond the open end of a semi-infinite channel is discussed. A succession of diffracted waves is created at the open end for which a general formula is obtained, by operational methods, when the pulse originates inside the channel. With the aid of a simple reciprocity relation the asymptotic behaviour of these diffracted waves can be used to deduce the form of the wave returning along the channel when the original pulse approaches the open end from an arbitrary direction.

Ultimately the returning wave becomes sensibly plane and separates into regions of length equal to the width of the channel, the form of the potential depending on the number of diffracted waves which contribute to each particular region. Explicit expressions are obtained for the potential in the first two regions at the head of the returning wave and for the third region when the pulse originates inside the channel.

The case of an initial velocity distribution given by the Heaviside unit pulse is treated in detail.

Introduction

Two problems are considered in the course of the paper and, for ease of reference, are designated problems I and II. Problem I is concerned with the behaviour of a transient sound wave or pulse progressing in the negative x-direction between the two bounding planes $y = 0$, $x \geqslant 0$ and $y = -2b$, $x \geqslant 0$ (figure 1). Physically, the wave originates inside a two-dimensional, semi-infinite channel, and is advancing towards the open end where it suffers reflexion and transmission. The asymptotic behaviour of the reflected and transmitted waves is obtained from a study of the diffraction effects which occur when the incident wave reaches, and progresses beyond, the open end.

In the second problem, the behaviour inside the channel is considered when the incident wave is external to the channel and approaches the open end from an arbitrary direction.

The first important step in the accurate solution of diffraction problems was made by Sommerfeld (1894, 1901) in his papers on the diffraction of waves by a semi-infinite wall. Various writers have since discussed this problem, but it is not until recently that major developments in the solution of a more complicated class of diffraction problems have taken place.

Levine & Schwinger (1948) consider the radiation of harmonic sound waves from a semi-infinite, circular pipe, essentially by reducing the mathematics to an integral equation susceptible to rigorous analytical solution.

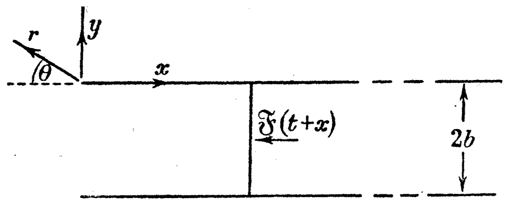

FIGURE 1

In the case of transient waves Fox (1948) has obtained the surface-pressure distribution on an infinitely long strip subjected to a sound pulse. Using operational methods, and with the help of Kirchhoff's solution of the wave equation, Fox also obtains an integral equation of a type similar to that discussed by Levine & Schwinger. Unfortunately the analysis becomes somewhat complicated if an attempt is made to calculate the whole pressure field at points off the surface of the strip by this method.

To overcome this difficulty in the present paper a technique first used by Gunn (1948) was employed, in which the potential is expressed directly in terms of its known boundary values with the aid of the appropriate Green's function. Although Gunn was primarily concerned with linearized supersonic aerofoil theory, the fundamental equations and boundary conditions are similar to those which occur in transient wave diffraction and his methods may be applied quite generally to problems in diffraction if the appropriate Green's function can be found.

The operational analogue of Sommerfeld's solution for diffraction by a semi-infinite wall is first obtained, and it is then shown how the subsequent diffracted waves, which appear due to the boundary conditions imposed by the second diffracting wall of the channel, can be built up from this initial solution. A reciprocity relation is also found which enables the solution of problem II to be derived from that of problem I.

Problem I. Pulse originating inside the channel

Before we become immersed in the mathematics of the problem, it is instructive to consider the behaviour of the pulse from a physical standpoint.

Suppose, then, a sound wave or pulse $\mathfrak{F}(t+x)\,H(t+x)$, where

$$\left.\begin{aligned} H(x) &= 1 \quad (x>0) \\ &= 0 \quad (x<0) \end{aligned}\right\}$$

is the Heaviside unit symbol, is proceeding along the channel in the negative x-direction (figure 1) and reaches the open end at time $t = 0$. (For economy of notation the sonic velocity is taken to be unity.)

Since any deviation from the relatively steady motion of the wave existing for $t<0$ is propagated with finite (sonic) velocity, the abrupt termination of the channel will, for small positive t, affect only those regions in the neighbourhood of each lip, and that part of the wave in the centre of the channel will not be modified. The potential is therefore considered in two parts. The original wave is allowed to propagate unchanged in the region between the extensions of the channel walls, so that at time t this part of the potential will affect the rectangular region bounded by $x=-t$, $y=0$, $y=-2b$. In addition, the knowledge that the channel has terminated will be propagated from each lip and gives rise to a further term in the form of two diffracted waves F_1 (figure 2), which will cancel the discontinuity introduced by the original wave along the extensions of the channel walls. The problem at this stage is effectively that of the diffraction of a sound pulse travelling parallel to a semi-infinite wall, since the appearance of F_1 at the upper lip is quite independent of the behaviour at the lower wall. The solution was first given by Sommerfeld (1901).

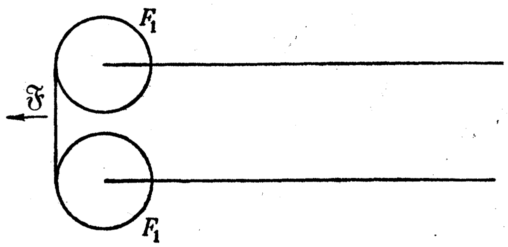

FIGURE 2

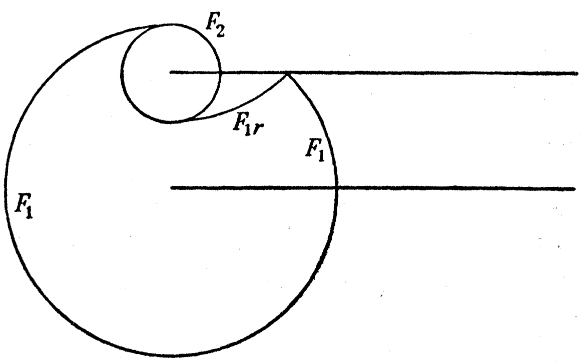

FIGURE 3

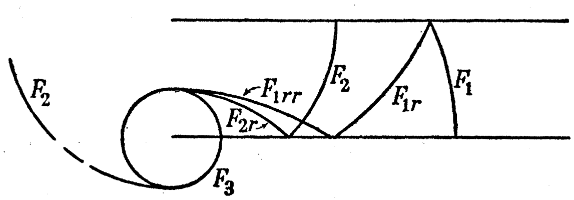

FIGURE 4

When F_1 is obtained, the above configuration (figure 2) is the complete solution to the problem during the time $0 \leqslant t \leqslant 2b$, where $2b$ is the width of the channel. At time $t=2b$ these first diffracted waves will reach the opposite sides and, during the interval $2b \leqslant t \leqslant 4b$, the potential must again be modified to satisfy the boundary conditions there.

That part of the wave outside the channel, and the part inside not having reached the opposite wall, will continue to propagate without modification. On reaching the wall at any point except the extreme edge, the wave suffers a reflexion in precisely the same manner as for a doubly infinite wall and the addition to the potential is that due to the image of the original wave in the wall. Finally, to preserve the continuity of the potential along the y-axis, we must add a further correcting term in the form of a second diffracted wave, starting out from the lip of the channel at time $2b$, as shown in figure 3. F_{1r} denotes the reflexion of F_1 in the upper wall, F_2 the second diffracted wave. For simplicity the configuration at the lower wall is omitted.

As t increases there will be further reflexions and diffractions and the potential becomes successively more complicated. The same principle, however, is applied of allowing each wave, or set of waves, to expand without modification unless they encounter one of the channel walls where they are simply reflected. In addition, each time a wave encounters the lip opposite to that from which it originates, a further diffracted wave appears. The procedure is essentially an iterative one; each diffracted wave, as it expands, gives rise to the next one in the series and subsequently suffers complete reflexion each time it encounters the bounding walls. Figure 4, for example, shows the configuration at the lower wall during the interval $2b \leqslant t \leqslant 4b$.

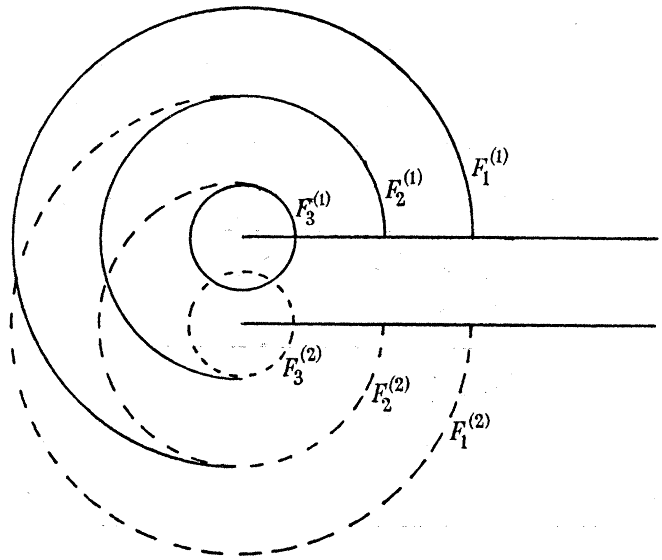

FIGURE 5

Inside the channel, and at large distances from the mouth, each set of successive reflexions will have an envelope consisting of a plane wave front travelling parallel to the walls with sonic velocity, and the potential will be a function only of $(t-x)$, say $\mathfrak{B}(t-x)$. Since the diffracted waves begin to propagate at regular intervals of time $2b$, it follows that the consecutive wave fronts are separated by a length equal to the width of the channel, and $\mathfrak{B}(t-x)$ may therefore be represented by the series

$$\begin{aligned}\mathfrak{B}(t-x) &= B_1(t-x)\,H(t-x)+B_2(t-x)\,H(t-x-2b)+\dots\\ &= \sum_{n=0}^{\infty} B_{n+1}(t-x)\,H(t-x-2nb),\end{aligned} \tag{1}$$

where B_n represents the asymptotic behaviour of the nth diffracted waves and their reflexions.

Outside the channel each diffracted wave, as it appears, will continue to expand in all directions except in the region of shadow bounded by the y-axis and the wall of the channel opposite to that from which it originates, where it does not penetrate (figure 5). Since the potential must be continuous, it follows that all the diffracted waves, except the first pair, have a discontinuity along the y-axis on one side of the channel and terminate abruptly on the other. There is also a discontinuity in each of the first diffracted waves, but this occurs along the extensions to the channel walls and cancels that which would otherwise be produced by the original wave $\mathfrak{F}$ as it proceeds beyond the mouth of the channel.

In order that the transmitted energy should remain finite its asymptotic value outside the channel must be of the form $\mathfrak{f}\{\theta, (t-r)\}/r^{\frac{1}{2}}$, where the co-ordinates (r, θ) are referred to an origin at the upper lip of the channel and such that the upper and lower sides of the channel wall are $\theta = \pi$, $\theta = -\pi$ respectively (figure 1). This is the asymptotic potential of a source, the strength of which is a function of θ. (The plane wave $\mathfrak{F}(t+x)$ is, at large distances, sufficiently annulled by the two waves $F_1^{(1)}$ and $F_1^{(2)}$ for the above formula also to hold for $\theta = 0$.)

Problem II. Pulse originating outside the channel

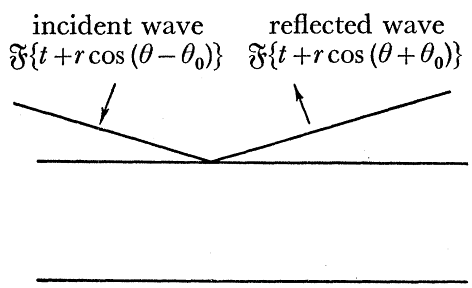

Figure 6

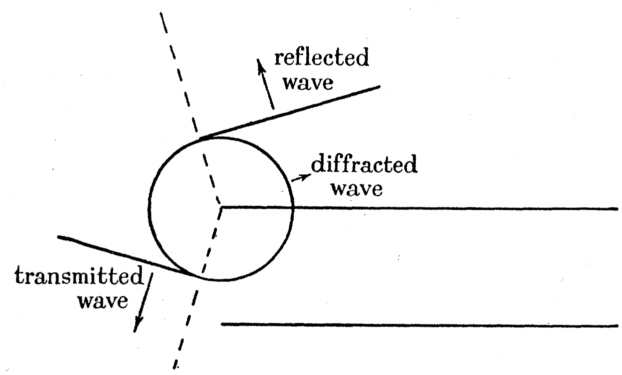

Figure 7

The behaviour of a plane wave approaching the channel from outside is, in many respects, similar to the previous problem. The wave is assumed to reach the upper lip at time $t = 0$ from a direction $\theta_0 > \frac{1}{2}\pi$. Then, during the interval $0 \leqslant t \leqslant 2b$, a transmitted wave

$$\mathfrak{F}\{t+r\cos(\theta-\theta_0)\}$$

progresses unchanged to the left of the line $\theta = -(\pi-\theta_0)$ and the reflected wave

$$\mathfrak{F}\{t+r\cos(\theta+\theta_0)\}$$

continues to the right of $\theta = (\pi-\theta_0)$. In addition there is a diffracted wave emanating from the upper lip and cancelling the discontinuities in potential along $\theta = \pm(\pi-\theta_0)$.

Thereafter successive diffractions and reflexions occur in a similar manner to those produced by a plane wave originating inside the channel with the modification that the diffracted waves $F_1, F_3, F_5, \ldots$ originate only at the upper lip, while $F_2, F_4, F_6, \ldots$ originate at the lower lip. The cumulative effect at large distances inside the channel is still that of a plane wave, and the envelopes of each set of reflexions are again separated by a length $2b$.

It is convenient at this point to remark that for any diffracted wave formed at the end of a semi-infinite wall, the velocity potential will be an odd function of θ. This can be seen by considering the effect of an incident pulse together with its image. The boundary condition of zero normal velocity along the wall will then be satisfied automatically and so no diffracted wave will be produced. The effect of the image is, then, equal and opposite to that of the original pulse and the result follows.

It follows that when the external wave is travelling along the upper walls of the channel $(\theta_0 = \pi)$, the potential of the successive diffracted waves produced at the open end will be similar to that produced by a wave originating inside the channel, but opposite in sign.

Since in the former case a single set of diffracted waves is produced, $F_1, F_3, F_5, \ldots$ at the upper lip, $F_2, F_4, F_6, \ldots$ at the lower lip, and in the latter case a double set, $F_1, F_2, F_3, \ldots$ at each lip, the potential in the waves returning along the channel in the two cases will, asymptotically, be equal and opposite apart from a factor $\frac{1}{2}$.

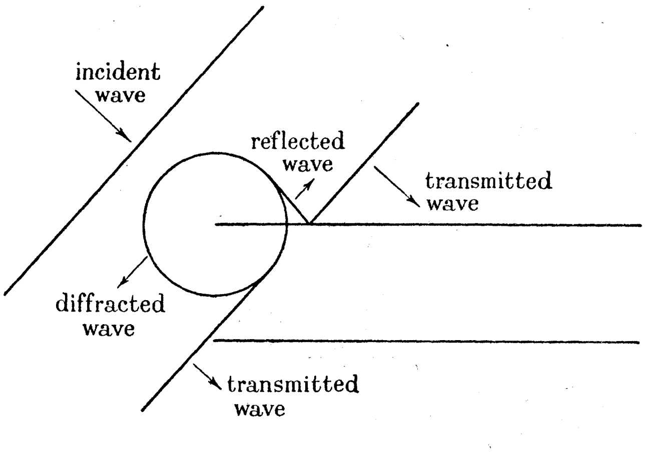

FIGURE 8

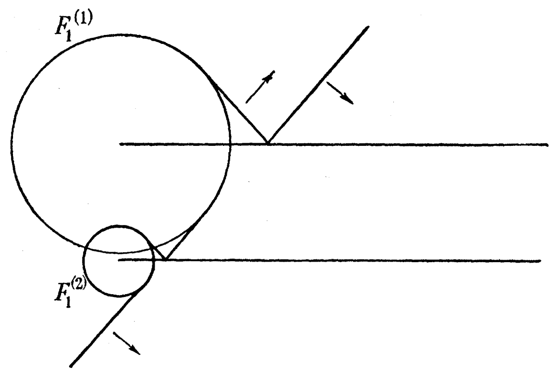

FIGURE 9

When $0<\theta_0<\frac{1}{2}\pi$ the behaviour is somewhat more complicated. At $t=0$ a diffracted wave $F_1^{(1)}$ begins to propagate at the upper lip, but the transmitted wave now encounters the lower wall at a time $t=2b\sin\theta_0$ and gives rise to a similar diffracted wave $F_1^{(2)}$. Thus the contributions to the potential inside the channel are as follows:

(i) That part of the original pulse which is propagated into the channel, together with its subsequent reflexions at the two walls. When the returning wave has become sensibly plane this will not contribute to the potential in the vicinity of the wave front (given by $x=t$), since it is propagated along the channel with velocity $\sin\theta_0$.

(ii) The reflected and diffracted waves arising from $F_1^{(1)}$ whose effects at large distances from the mouth are simulated by the appropriate series of plane waves, the successive wave fronts being separated by a length $2b$.

(iii) The reflected and diffracted waves arising from $F_1^{(2)}$. Since the diffracted waves are all odd functions of θ, and in particular $F_1^{(1)}$ and $F_1^{(2)}$ are anti-symmetrical with respect to the centre-line (apart from a time-lag $2b\sin\theta_0$), it follows that this contribution is equal and opposite to that due to the reflexions and diffractions of $F_1^{(1)}$ if the appropriate correction is made to the time factor.

The potential at infinity is therefore given by two equal and opposite plane waves superimposed with their fronts separated by a length $2b\sin\theta_0$, and if $\mathfrak{C}(t-x,\theta_0)$ is the limiting potential, then

$$\left.\begin{aligned}\mathfrak{C}(t-x,\theta_0) &= \sum_{n=0}^{\infty} C_{n+1}(t-x,\theta_0)\,H(t-x-2nb) \quad (\tfrac{1}{2}\pi<\theta_0<\pi)\\ &= \sum_{n=0}^{\infty} C_{n+1}(t-x,\theta_0)\,H(t-x-2nb)-C_{n+1}(t-x-2b\sin\theta_0,\theta_0)\\ &\qquad\times H(t-x-2b\sin\theta_0-2nb) \quad (0<\theta_0<\tfrac{1}{2}\pi).\end{aligned}\right\} \tag{2}$$

Diffraction by a semi-infinite plane

The first step in the mathematical solution of the problem is to obtain the diffraction effects when a sound wave encounters a semi-infinite plane boundary. When the wave is travelling parallel to the boundary this, as previously stated, is effectively the solution to problem I until the first diffracted waves expand sufficiently to suffer modification themselves, since the behaviour at the upper wall, which produces $F_1^{(1)}$, is quite independent of that at the lower wall.

In this section a general formula is obtained for the diffraction of a wave approaching the boundary from an arbitrary direction θ_0, since this result will be required in deducing the general asymptotic behaviour of the potential in problem II.

The final potential χ (say) will be a solution of

$$\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}=\frac{\partial^2\chi}{\partial t^2}, \tag{3}$$

subject to the condition of zero normal derivative along the boundary.

The analysis is carried out in terms of the operational calculus, so that equation (3) becomes

$$\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}=p^2\chi \tag{4}$$

(see Jeffreys & Jeffreys 1946, and appendix 3).

For negative values of t, the contribution to χ is known to be an incident pulse

$$\mathfrak{F}\{t+r\cos(\theta-\theta_0)\}\,H\{t+r\cos(\theta-\theta_0)\},$$

and its reflexion $\quad \mathfrak{F}\{t+r\cos(\theta+\theta_0)\}\,H\{t+r\cos(\theta+\theta_0)\}\quad$ if $\quad\theta_0>\frac{1}{2}\pi.$

Operationally, these contributions become

$$\exp[pr\cos(\theta-\theta_0)]\,F(p)\quad\text{and}\quad\exp[pr\cos(\theta+\theta_0)]\,F(p)$$

respectively, where

$$\mathfrak{F}(t)=F(p). \tag{5}$$

For $t>0$ these contributions persist in the regions $-(\pi-\theta_0)<\theta<\pi$ and $(\pi-\theta_0)<\theta<\pi$ respectively (figures 7 and 9). In addition, there is a diffracted pulse emanating from the lip.

Since the function χ now satisfies the equation $(\nabla^2-p^2)\,\chi=0$, the problem is analogous to that of the diffraction of a plane harmonic wave $\exp[-ikr\cos(\theta-\theta_0)]$ and can be solved in a like manner. The solution of the harmonic problem, first given by Sommerfeld (1894) and later discussed by Carslaw (1899) and Lamb (1907), is

$$\left(\frac{k}{i\pi}\right)^{\frac{1}{2}}\left\{\exp[-ikr\cos(\theta-\theta_0)]\int_{-\infty}^{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}e^{ik\lambda^2}\,d\lambda\right.$$
$$\left.+\exp[-ikr\cos(\theta+\theta_0)]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}^{\infty}e^{ik\lambda^2}\,d\lambda\right\}\quad(-\pi<\theta<\pi). \tag{6}$$

Since the operational form of the incident pulse is $\exp[pr\cos(\theta-\theta_0)]\,F(p)$, the solution can be obtained, formally, by substituting ip for k in expression (6) and inserting the factor F, so that

$$\chi=F\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left\{\exp[pr\cos(\theta-\theta_0)]\int_{-\infty}^{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}e^{-p\lambda^2}\,d\lambda\right.$$
$$\left.+\exp[pr\cos(\theta+\theta_0)]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}^{\infty}e^{-p\lambda^2}\,d\lambda\right\}\quad(-\pi<\theta<\pi). \tag{7}$$

The above result could be obtained directly by an analysis parallel to that used by Sommerfeld, but it is easily verified that χ satisfies all the conditions of the problem. Thus $\partial\chi/\partial\theta$ is zero along the two sides of the plane ($\theta=\pm\pi$), and the relation

$$\int_{-\infty}^{\infty} e^{-p\lambda^2}\,d\lambda=\left(\frac{\pi}{p}\right)^{\frac{1}{2}} \tag{8}$$

shows that χ contains the relevant plane-wave contributions, $\exp[pr\cos(\theta-\theta_0)]F$ for $-(\pi-\theta_0)<\theta<\pi$ and $\exp[pr\cos(\theta+\theta_0)]F$ for $(\pi-\theta_0)<\theta<\pi$, together with a correction term which is $O(r^{-\frac{1}{2}})$ for large r and fixed θ except along the lines $\theta=\pm(\pi-\theta_0)$. This last peculiarity of the solution is essential if the potential is to remain continuous.

The diffracted wave F_1 is obtained from χ by subtracting the plane-wave contributions from the appropriate intervals, thus

$$\left.\begin{aligned}
F_1 &= -F\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left\{\exp[pr\cos(\theta-\theta_0)]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda+\exp[pr\cos(\theta+\theta_0)]\int_{-\infty}^{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)} e^{-p\lambda^2}\,d\lambda\right\}\\
&\qquad\qquad [(\pi-\theta_0)<\theta<\pi]\\
&= -F\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left\{\exp[pr\cos(\theta-\theta_0)]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda-\exp[pr\cos(\theta+\theta_0)]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda\right\}\\
&\qquad\qquad [-(\pi-\theta_0)<\theta<(\pi-\theta_0)]\\
&= F\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left\{\exp[pr\cos(\theta-\theta_0)]\int_{-\infty}^{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)} e^{-p\lambda^2}\,d\lambda+\exp[pr\cos(\theta+\theta_0)]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda\right\}\\
&\qquad\qquad [-\pi<\theta<-(\pi-\theta_0)].
\end{aligned}\right\} \tag{9}$$

To interpret this result we first note that

$$\operatorname{sgn}[\cos\tfrac{1}{2}\theta]\frac{p^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\exp[pr\cos\theta]\int_{(2r)^{\frac{1}{2}}|\cos\frac{1}{2}\theta|}^{\infty} e^{-p\lambda^2}\,d\lambda=\frac{\cos\frac{1}{2}\theta\,H(t-r)}{2^{\frac{1}{2}}\pi(t/r-1)^{\frac{1}{2}}\{t+r\cos\theta\}}. \tag{10}$$

Moreover, by the principle of superposition,

$$\frac{F(p)\,G(p)}{p}=\int_0^t \mathfrak{F}(t-\tau)\,\mathfrak{G}(\tau)\,d\tau, \tag{11}$$

where $F(p)$, $G(p)$ are the operational forms of the functions $\mathfrak{F}(t)$, $\mathfrak{G}(t)$ appearing on the right-hand side of equation (11). It follows immediately that

$$\begin{aligned}
F_1 &= -\frac{\cos\frac{1}{2}(\theta-\theta_0)}{2^{\frac{1}{2}}\pi}\int_r^t \frac{\mathfrak{F}(t-\tau)\,d\tau}{(\tau/r-1)^{\frac{1}{2}}\{\tau+r\cos(\theta-\theta_0)\}}+\frac{\cos\frac{1}{2}(\theta+\theta_0)}{2^{\frac{1}{2}}\pi}\int_r^t \frac{\mathfrak{F}(t-\tau)\,d\tau}{(\tau/r-1)^{\frac{1}{2}}\{\tau+r\cos(\theta+\theta_0)\}}\\
&= -\frac{(2r)^{\frac{1}{2}}}{\pi}\Bigg[\cos\tfrac{1}{2}(\theta-\theta_0)\int_0^{(t-r)^{\frac{1}{2}}}\frac{\mathfrak{F}(t-r-z^2)\,dz}{\{z^2+2r\cos^2\frac{1}{2}(\theta-\theta_0)\}}\\
&\qquad\qquad -\cos\tfrac{1}{2}(\theta+\theta_0)\int_0^{(t-r)^{\frac{1}{2}}}\frac{\mathfrak{F}(t-r-z^2)\,dz}{\{z^2+2r\cos^2\frac{1}{2}(\theta+\theta_0)\}}\Bigg].
\end{aligned} \tag{12}$$

When the original wave travels parallel to the wall ($\theta_0=\pm\pi$), the above expression holds only if the correct form for the potential of the original wave is used. Reference to figure 6 shows that, as $\theta_0\to\pi$, the incident wave and its reflexion coalesce to give a resultant potential $2\mathfrak{F}(t-r\cos\theta)$. Thus the limiting form of equation (12) is, in effect, the potential of a diffracted wave produced by an incident pulse $2\mathfrak{F}(t-r\cos\theta)$ travelling parallel to the wall. If the incident pulse is $\mathfrak{F}(t-r\cos\theta)$, the correct solution, for $\theta_0=\pi$, will be

$$F_1 = \left.\begin{array}{ll} -F\left(\dfrac{p}{\pi}\right)^{\frac{1}{2}}\exp[-pr\cos\theta]\displaystyle\int_{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} e^{-p\lambda^2}\,d\lambda & (0<\theta<\pi) \\ F\left(\dfrac{p}{\pi}\right)^{\frac{1}{2}}\exp[-pr\cos\theta]\displaystyle\int_{-\infty}^{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\theta} e^{-p\lambda^2}\,d\lambda & (-\pi<\theta<0) \end{array}\right\} \tag{13}$$

$$= -\frac{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\theta}{\pi}\int_0^{(t-r)^{\frac{1}{2}}}\frac{\mathfrak{F}(t-r-z^2)\,dz}{\{z^2+2r\sin^2\frac{1}{2}\theta\}} \quad (-\pi<\theta<\pi). \tag{14}$$

The unit pulse, in which the velocity is given by the Heaviside unit function, offers a simple illustration of the above formulae. The potential of the incident pulse is given by

$$\mathfrak{F}\{t+r\cos(\theta-\theta_0)\} = -\{t+r\cos(\theta-\theta_0)\}\,H\{t+r\cos(\theta-\theta_0)\},$$

so that

$$F_1 = \frac{(2r)^{\frac{1}{2}}}{\pi}\left[\cos\tfrac{1}{2}(\theta-\theta_0)\int_0^{(t-r)^{\frac{1}{2}}}\frac{t-r-z^2}{z^2+2r\cos^2\frac{1}{2}(\theta-\theta_0)}\,dz - \cos\tfrac{1}{2}(\theta+\theta_0)\int_0^{(t-r)^{\frac{1}{2}}}\frac{t-r-z^2}{z^2+2r\cos^2\frac{1}{2}(\theta+\theta_0)}\,dz\right]$$

$$= \frac{1}{\pi}\left[\{t+r\cos(\theta-\theta_0)\}\tan^{-1}\left[\frac{(t-r)^{\frac{1}{2}}}{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}\right] - (2r)^{\frac{1}{2}}(t-r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\theta_0)\right.$$

$$\left. -\{t+r\cos(\theta+\theta_0)\}\tan^{-1}\left[\frac{(t-r)^{\frac{1}{2}}}{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}\right] + (2r)^{\frac{1}{2}}(t-r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta+\theta_0)\right]$$

$$(-\pi<\theta<\pi;\ -\pi<\theta_0<\pi), \tag{15}$$

whereas, for $\theta_0=\pi$, the analogous expression is

$$F_1 = \frac{1}{\pi}\left[(t-r\cos\theta)\tan^{-1}\left[\frac{(t-r)^{\frac{1}{2}}}{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\theta}\right] - (2r)^{\frac{1}{2}}(t-r)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta\right]. \tag{16}$$

Reciprocity relation

With the aid of F_1, the potential in the first region at the head of the returning wave could be obtained directly by setting up the appropriate image series, as indicated above, and calculating its asymptotic behaviour. This, however, is rendered unnecessary, and the subsequent analysis is greatly simplified by an interesting and useful reciprocity relation between the asymptotic value of the potential resulting from a pulse $\mathfrak{F}(t+x)$ originating inside the channel, and a similar pulse $\mathfrak{F}\{t+r\cos(\theta-\theta_0)\}$ approaching the channel from a direction θ_0.

If the final potential fields, after diffraction, resulting from two such pulses are denoted by ϕ_a, ϕ_b respectively, then the reciprocity relation connects the asymptotic behaviour of ϕ_a outside the channel with that of ϕ_b inside the channel.

An analogous relation for the case of a harmonic wave propagating in the presence of a circular pipe has been given by Levine & Schwinger (1948).

It has already been observed that ϕ_a behaves like $r^{-\frac{1}{2}}\mathfrak{f}(t-r,\theta)$ at large distances outside the channel. However, whereas in the case of ϕ_a the plane wave $\mathfrak{F}(t+x)$ which continues to propagate from the channel mouth is, at large distances, sufficiently annulled by the two waves $F_1^{(2)}$, $F_1^{(2)}$ for the above relation to be true when $\theta=0$, the plane-wave contributions in ϕ_b must be retained. Since the reflected and transmitted waves are propagated in the regions $(\pi-\theta_0)<\theta<\pi$ and $-(\pi-\theta_0)<\theta<\pi$ respectively, it follows that the correction term, which must maintain the continuity of the potential in the transition between the two regions, cannot tend to zero in the directions $\theta=\pm(\pi-\theta_0)$. Similar remarks apply to diffraction by a semi-infinite barrier, and reference to equation (7) shows that, asymptotically, the potential consists of the relevant plane-wave contributions together with a term which is $O(r^{-\frac{1}{2}})$ for large r and fixed θ except along $\theta=\pm(\pi-\theta_0)$. In the case of the two-dimensional channel the asymptotic behaviour of the potential will be essentially similar to that produced by a semi-infinite barrier. Owing to the presence of the second diffracting wall there will be further diffraction effects, but it will be seen that all the diffracted waves, except F_1, are asymptotically $O(r^{-\frac{1}{2}})$ uniformly in θ.

Thus, with the help of equation (7), the asymptotic behaviour of ϕ_b may be written as

$$\phi_b \sim F\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left\{\exp\left[pr\cos(\theta-\theta_0)\right]\int_{-\infty}^{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)} e^{-p\lambda^2}\,d\lambda \right.$$
$$\left. +\exp\left[pr\cos(\theta+\theta_0)\right]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda\right\}+g(\theta)\frac{e^{-pr}}{r^{\frac{1}{2}}}. \qquad (17)$$

The function $g(\theta)$ will also contain the parameter p.

When $0<\theta_0<\frac{1}{2}\pi$, the exact form of ϕ_b should take account of the incident pulse in a region to the left of a line parallel to $\theta=-(\pi-\theta_0)$ but passing through the lower lip of the channel (see figure 9). It can be shown, however, that even in this case equation (17) still represents the asymptotic form of ϕ_b.

The functions ϕ_a, ϕ_b have, then, the following asymptotic behaviour:

inside the channel:

$$\left.\begin{aligned}\phi_a &\sim F e^{px}+B e^{-px},\\ \phi_b &\sim \qquad\quad C e^{-px};\end{aligned}\right\} \qquad (18)$$

outside the channel:

$$\left.\begin{aligned}&\phi_a \sim f(\theta)\frac{e^{-pr}}{r^{\frac{1}{2}}},\\ &\phi_b \sim F\exp\left[pr\cos(\theta-\theta_0)\right]+g(\theta)\frac{e^{-pr}}{r^{\frac{1}{2}}}-F\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left\{\exp\left[pr\cos(\theta-\theta_0)\right]\right.\\ &\quad\times\left.\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda-\exp\left[pr\cos(\theta+\theta_0)\right]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)}^{\infty} e^{-p\lambda^2}\,d\lambda\right\};\end{aligned}\right\} \qquad (19)$$

where $B(p)$, $C(p)$, $f(p,\theta)$, $g(p,\theta)$ are the operational representations of $\mathfrak{B}(t)$, $\mathfrak{C}(t)$, $\mathfrak{f}(t,\theta)$, $\mathfrak{g}(t,\theta)$ respectively.

The reciprocity relation is obtained by applying Green's theorem:

$$\iint_S \{\phi_a\nabla^2\phi_b-\phi_b\nabla^2\phi_a\}\,dS=\int_s\left\{\phi_a\frac{\partial\phi_b}{\partial n}-\phi_b\frac{\partial\phi_a}{\partial n}\right\}ds \qquad (20)$$

to the functions ϕ_a, ϕ_b, in the region bounded by the curves s_1, s_2, s_3 shown in figure 10. s_3 is a large circle, centre A, the origin of the (r,θ) co-ordinates.

The left-hand side of (20) vanishes since both ϕ_a and ϕ_b satisfy $(\nabla^2-p^2)\phi=0$. Their normal derivatives also vanish along s_2, so that

$$\int_{s_1}\left\{\phi_a\frac{\partial\phi_b}{\partial n}-\phi_b\frac{\partial\phi_a}{\partial n}\right\}ds+\int_{s_3}\left\{\phi_a\frac{\partial\phi_b}{\partial n}-\phi_b\frac{\partial\phi_a}{\partial n}\right\}ds=0. \qquad (21)$$

If s_1 and s_3 are at a sufficiently great distance from the mouth of the channel, the asymptotic forms of the two potentials may be used. The integral along s_1 then becomes

$$\int_{-b}^{b}\left[(F\,e^{px}+B\,e^{-px})\frac{\partial}{\partial x}(C\,e^{-px})-C\,e^{-px}\frac{\partial}{\partial x}(F\,e^{px}+B\,e^{-px})\right]_{x=-L}dy$$
$$=-p\int_{-b}^{b}[C\,e^{-pL}\{F\,e^{pL}+B\,e^{-pL}\}+C\,e^{-pL}\{F\,e^{pL}-B\,e^{-pL}\}]\,dy$$
$$=-4bpCF. \quad (22)$$

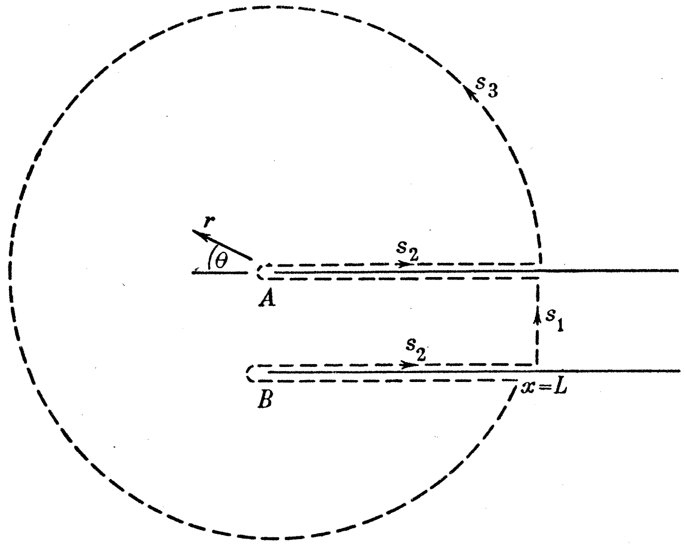

FIGURE 10

In the evaluation of the integral along s_3, it should be noted that while ϕ_a contains a factor e^{-pr}, the only exponential factors occurring in ϕ_b, whose exponents are capable of taking positive values on s_3, are of the form $\exp[pr\cos(\theta-\theta_0)]$ and $\exp[pr\cos(\theta+\theta_0)]$.

Moreover, as $r\to\infty$

$$\exp[pr\cos\theta]\int_{(2r)^{\frac{1}{2}}\cos\frac{1}{2}\theta}^{\infty}e^{-p\lambda^2}\,d\lambda\sim\frac{e^{-pr}}{2p(2r)^{\frac{1}{2}}\cos\frac{1}{2}\theta} \quad (23)$$

if $\cos\theta/2\neq 0$.

It follows that, in the limit as $r\to\infty$, the only contribution to the integral along s_3 will be from the term $F\exp[pr\cos(\theta-\theta_0)]$, the integration being confined to a small interval $|\theta-\theta_0|<\epsilon$.

Thus

$$\lim_{r\to\infty}\int_{s_3}\left\{\phi_a\frac{\partial\phi_b}{\partial n}-\phi_b\frac{\partial\phi_a}{\partial n}\right\}ds$$
$$=\lim_{r\to\infty}\int_{\theta_0-\epsilon}^{\theta_0+\epsilon}\left[f(\theta)\frac{e^{-pr}}{r^{\frac{1}{2}}}\frac{\partial}{\partial r}\{F\exp[pr\cos(\theta-\theta_0)]\}-F\exp[pr\cos(\theta-\theta_0)]\frac{\partial}{\partial r}\left(f(\theta)\frac{e^{-pr}}{r^{\frac{1}{2}}}\right)\right]r\,d\theta$$
$$=\lim_{r\to\infty}pFr^{\frac{1}{2}}\int_{\theta_0-\epsilon}^{\theta_0+\epsilon}f(\theta)\,[1+\cos(\theta-\theta_0)]\exp[-pr\{1-\cos(\theta-\theta_0)\}]\,d\theta,$$

and, since the interval is arbitrarily small, we may write this, in the limit, as

$$\lim_{r\to\infty}2r^{\frac{1}{2}}pFf(\theta_0)\int_{-\epsilon}^{\epsilon}\exp[-2pr\sin^2\tfrac{1}{2}\eta]\cos\tfrac{1}{2}\eta\,d\eta$$
$$=2^{\frac{3}{2}}pFf(\theta_0)\lim_{r\to\infty}\int_{-(2r)^{\frac{1}{2}}\sin\frac{1}{2}\epsilon}^{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\epsilon}e^{-p\xi^2}\,d\xi$$
$$=2^{\frac{3}{2}}\pi^{\frac{1}{2}}p^{\frac{1}{2}}Ff(\theta_0). \quad (24)$$

Substitution of (22) and (24) in (21) gives

$$C = \frac{1}{b}\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} f(\theta_0) \quad (0<\theta_0<\pi). \tag{25}$$

Again, in the limit $\theta_0 = \pi$, this holds for an incident wave $2\mathfrak{F}(t-r\cos\theta)$ outside the channel. If the incident wave is $\mathfrak{F}(t-r\cos\theta)$, then

$$C(\pi) = \frac{1}{2b}\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} f(\pi). \tag{26}$$

It has already been stated that the diffraction effects of two similar plane waves, which approach the mouth from inside the channel and from a direction $\theta_0 = \pi$ respectively, are such that the waves which return along the channel in the two cases are equal and opposite apart from a factor $\frac{1}{2}$, so that finally

$$B = -\frac{1}{b}\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} f(\pi). \tag{27}$$

Equations (25), (26) and (27) constitute the reciprocity relation, and offer a means of calculating the absorbed wave from any incident pulse once the value of $f(\theta)$ is known. Problems I and II are therefore reduced to an investigation of the asymptotic behaviour outside the channel of the various diffracted waves resulting from an incident pulse originating inside the channel.

Contribution from the first diffracted waves

Since the first diffracted wave has already been obtained, its contribution to $f(\theta)$, and hence the first terms in the expansion of C and B, can now be calculated.

For the case of an incident wave originating inside the channel, the value of F_1 for the particular case $\theta_0 = -\pi$ is required. Its operational form is readily deduced from equation (13):

$$\left.\begin{aligned} F_1 &= F\left(\frac{p}{\pi}\right)^{\frac{1}{2}} \exp\left[-pr\cos\theta\right] \int_{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} e^{-p\lambda^2}\,d\lambda \quad (0<\theta<\pi) \\ &= -F\left(\frac{p}{\pi}\right)^{\frac{1}{2}} \exp\left[-pr\cos\theta\right] \int_{-\infty}^{(2r)^{\frac{1}{2}}\sin\frac{1}{2}\theta} e^{-p\lambda^2}\,d\lambda \quad (-\pi<\theta<0). \end{aligned}\right\} \tag{28}$$

Such a wave will emanate from each lip of the channel, $F_1^{(1)}$ being given by (28) and $F_1^{(2)}$ being deduced from the fact that the resulting potential from the two waves is symmetrical about the centre line of the channel. Thus, considering the interval $0<\theta<\pi$ and remembering that $F_1^{(2)}$ does not penetrate into the region $\frac{1}{2}\pi<\theta<\pi$, we find that the contribution to the potential outside the channel is

$$\left.\begin{aligned} F_1^{(1)}+F_1^{(2)} &\quad (0<\theta<\tfrac{1}{2}\pi), \\ F_1^{(1)} &\quad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{29}$$

Their asymptotic forms, referred to an origin at the upper lip of the channel, are

$$\left.\begin{aligned} F_1^{(1)}+F_1^{(2)} &\sim \frac{F\,e^{-pr}}{2(2p\pi r)^{\frac{1}{2}}\sin\frac{1}{2}\theta}\left(1-\exp\left[-2pb\sin\theta\right]\right) \quad (0<\theta<\tfrac{1}{2}\pi), \\ F_1^{(1)} &\sim \frac{F\,e^{-pr}}{2(2p\pi r)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \quad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{30}$$

It follows immediately that the contribution to $f(\theta)$ from the first diffracted waves is $f_1(\theta)$ (say), where

$$\left.\begin{aligned} f_1(\theta) &= \frac{F(1-\exp[-2pb\sin\theta])}{2(2p\pi)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \quad (0<\theta<\tfrac{1}{2}\pi), \\ &= \frac{F}{2(2p\pi)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \quad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{31}$$

Substitution in equation (25) gives

$$C_1 = \frac{F}{4bp\sin\frac{1}{2}\theta_0} = \frac{1}{4b\sin\frac{1}{2}\theta_0}\int_0^t \mathfrak{F}(\tau)\,d\tau \quad (0<\theta_0<\pi). \tag{32}$$

The factor $(1-\exp[-2pb\sin\theta_0])$, occurring for $0<\theta_0<\frac{1}{2}\pi$, is, on referring to equation (2), precisely that required to produce the two terms

$$C_1(t-x,\theta_0)\,H(t-x) - C_1(t-x-2b\sin\theta_0,\theta_0)\,H(t-x-2b\sin\theta_0),$$

where, with the help of equation (32),

$$C_1(t-x,\theta_0) = \frac{1}{4b\sin\frac{1}{2}\theta_0}\int_0^{t-x}\mathfrak{F}(\tau)\,d\tau. \tag{33}$$

Finally, from (26) and (27),

$$2C_1(\pi) = -B_1 = \frac{F}{4bp} \tag{34}$$

or

$$B_1(t-x) = -\frac{1}{4b}\int_0^{(t-x)}\mathfrak{F}(\tau)\,d\tau. \tag{35}$$

These results may be checked directly if the effect of the successive reflexions is simulated by an appropriate image series along the y-axis. This was done for the particular case of the wave originating inside the channel, and it was found that the expression for the potential can be written

$$\phi_1 = -\frac{F\,\mathrm{e}^{-px}}{4bp} - \frac{2^{\frac{1}{2}}p^{\frac{1}{2}}F}{2b}\sum_{n=1}^{\infty}\frac{\exp\left[-x\left\{p^2+\left(\frac{n\pi}{b}\right)^2\right\}^{\frac{1}{2}}\right]\cos\left(\frac{n\pi y}{b}\right)}{\left\{p^2+\left(\frac{n\pi}{b}\right)^2\right\}^{\frac{1}{2}}\left[p+\left\{p^2+\left(\frac{n\pi}{b}\right)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}}. \tag{36}$$

The dominant term, when the wave has become sensibly plane, is the first, which is equivalent to $-\frac{1}{4b}\int_0^{t-x}\mathfrak{F}(\tau)\,d\tau$ and agrees with equation (35).

The behaviour of the correction term was also studied and it was found that, if the mth derivative of $\mathfrak{F}(z)$ has a finite number of discontinuities in the range $0\leqslant z\leqslant(t-x)$ (including the particular case $\mathfrak{F}^{(m)}(0)\neq 0$), while all lower derivatives are continuous, then the correction term is $O\{(t+x)^{-\frac{1}{2}(2m+3)}\}$.

Further diffracted waves

The first two diffracted waves are the complete solution of the diffraction problem in the interval $0\leqslant t\leqslant 2b$. For $t>2b$, however, $F_1^{(1)}$ and $F_1^{(2)}$ encounter the opposite walls and violate the condition of zero normal velocity. The solution can be continued if a wave can be found which, starting to propagate at time $t=2b$, annuls the normal velocity of the initial wave emanating from the opposite lip. Two such waves will be required, one for each lip of the channel. It is convenient to consider the actual correction to F_1 which, in the previous notation, will consist of F_2 together with the reflexion of F_1 inside the channel and a term which cancels F_1 itself in the region of shadow of the wall. The correction will then be a continuous solution of the wave equation except across the wall (F_2 being discontinuous along the y-axis). Moreover, in the calculation of this correction term at the upper wall, the lower wall may be ignored since it has no effect until time $t = 4b$.

Thus the problem consists in finding a function ψ_2 such that it

(i) satisfies the equation $\nabla^2\psi_2 = p^2\psi_2$,
(ii) is continuous for $-\pi < \theta < \pi$,
(iii) $\rightarrow 0$ at infinity, and
(iv) has a specified normal derivative along $\theta = \pm\pi$.

The solution, when obtained, is then valid for $2b \leqslant t \leqslant 4b$, beyond which the wave ψ_2 encounters the opposite wall and the process is repeated with the function ψ_2 assuming the role of F_1 and giving rise to a further correction term ψ_3. Such modification will be necessary after successive intervals of time $2b$ and each additional term can be obtained once its predecessor is known.

The above problem can be solved if the Green function for a semi-infinite barrier can be found. It is known from Green's theorem that if ψ and G are two solutions of the equation $(\nabla^2 - p^2)\,\phi = 0$, ψ and its derivatives being continuous inside and on a closed curve c, and G being continuous except at a point P where $G \sim -\log\rho$, ρ being the radial distance from P, then

$$\psi(P) = \frac{1}{2\pi}\int_c \left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) ds, \tag{37}$$

where n is the outward normal to the curve c. If, in addition, G is such that

$$\frac{\partial G}{\partial n} = 0 \quad \text{on} \quad c,$$

then

$$\psi(P) = \frac{1}{2\pi}\int_c G\frac{\partial\psi}{\partial n}\, ds, \tag{38}$$

which gives the value of ψ at all points inside c in terms of its normal derivative on c.

In the present problem the curve c will consist of the two sides of the wall ($\theta = \pm\pi$) together with a large circle whose radius tends to infinity.

Green's function for a semi-infinite barrier

The determination of the Green function satisfying the condition of zero normal derivative along $\theta = \pm\pi$ is, formally, analogous to the problem of the diffraction of a two-dimensional harmonic source of sound by a semi-infinite barrier. This problem has been solved by Carslaw (1899) using Sommerfeld's technique.

The derivation of the above and allied Green functions, and their application to problems in supersonic flow, has been fully discussed by Gunn (1948, p. 345), who also showed that the somewhat cumbersome expression obtained by Carslaw can be transformed into a single function valid over the whole range $-\pi < \theta < \pi$.

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In the present problem, Gunn's solution may be written as

$$G(r,\theta;\,r_0,\theta_0)=D(r,r_0,\overline{\theta-\theta_0})+D(r,r_0,\overline{\theta+\theta_0}), \tag{39}$$

where (r_0,θ_0) are the co-ordinates of the point P, and

$$D(r,r_0,\phi)=\frac{1}{2}\int_{-\infty}^{2(rr_0)^{\frac{1}{2}}\cos\frac{1}{2}\phi}\frac{\exp\left[-p(r^2+r_0^2-2rr_0\cos\phi+\tau^2)^{\frac{1}{2}}\right]d\tau}{(r^2+r_0^2-2rr_0\cos\phi+\tau^2)^{\frac{1}{2}}}. \tag{40}$$

Application of Green's function

With the help of G and the first diffracted wave, say $F_1^{(2)}$, the addition to the potential at the upper wall can now be calculated for $2b\leqslant t\leqslant 4b$.

The contour of integration consists of the two sides of the upper wall, together with a large circle whose radius tends to infinity. The integral round the circle will, in the limit, tend to zero so that, if $F_1^{(2)}(r,\theta)$ is referred to co-ordinates about the origin at the upper lip of the channel, the modification to the potential will be, from equation (38),

$$\begin{aligned}\psi_2(r_0,\theta_0)&=-\frac{1}{2\pi}\int_0^\infty\left\{\frac{\partial}{\xi\,\partial\theta}F_1^{(2)}(\xi,\theta)\right\}_{\theta=\pi}\{G(\xi,\pi;\,r_0,\theta_0)-G(\xi,-\pi;\,r_0,\theta_0)\}\,d\xi\\&=-\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\,\partial\theta}F_1^{(2)}(\xi,\theta)\right\}_{\theta=\pi}\{D(\xi,r_0,\overline{\pi-\theta_0})-D(\xi,r_0,\overline{\pi+\theta_0})\}\,d\xi\\&=-\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\,\partial\theta}F_1^{(2)}(\xi,\theta)\right\}_{\theta=\pi}d\xi\left\{\int_0^{2(\xi r_0)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0}\frac{\exp\left[-p(\xi^2+r_0^2+2\xi r_0\cos\theta_0+\tau^2)^{\frac{1}{2}}\right]d\tau}{(\xi^2+r_0^2+2\xi r_0\cos\theta_0+\tau^2)^{\frac{1}{2}}}\right\},\end{aligned} \tag{41}$$

or

$$\begin{aligned}\psi_2(r,\theta)=\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\,\partial\theta}F_1^{(2)}(\xi,\theta)\right\}_{\theta=\pi}d\xi\Big\{&K_0[p(r^2+\xi^2+2\xi r\cos\theta)^{\frac{1}{2}}]\\&-\int_{-\infty}^{2(r\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}\frac{\exp\left[-p(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}\right]d\tau}{(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}}\Big\},\end{aligned} \tag{42}$$

where the relation

$$K_0[pz]=\int_0^\infty\frac{\exp\left[-p(z^2+\tau^2)^{\frac{1}{2}}\right]d\tau}{(z^2+\tau^2)^{\frac{1}{2}}} \tag{43}$$

has been used. There will be a similar contribution at the lower lip due to $F_1^{(1)}$.

This process can be repeated at intervals of time $2b$; in general, if $\psi_n^{(2)}(r,\theta)$ is the correction which is added at the lower lip for $2(n-1)\,b\leqslant t\leqslant 2nb$, then

$$\begin{aligned}\psi_{n+1}^{(1)}(r,\theta)=\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi}d\xi\Big\{&K_0[p(r^2+\xi^2+2\xi r\cos\theta)^{\frac{1}{2}}]\\&-\int_{-\infty}^{2(r\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}\frac{\exp\left[-p(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}\right]d\tau}{(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}}\Big\}.\end{aligned} \tag{44}$$

Note that $\psi_{n+1}^{(1)}+\psi_{n+1}^{(2)}$ represents the *total* correction which must be added, for

$$2(n-1)\,b\leqslant t\leqslant 2nb,$$

to the expression for the potential previously valid for $t\leqslant 2(n-1)\,b$. The potential at all points both inside and outside the channel is, in fact, represented by the series

$$\phi_a=\sum_{n=1}^{\infty}\psi_n^{(1)}+\psi_n^{(2)}, \tag{45}$$

where $\psi_1=F_1$, but, for $n\geqslant 2$, ψ_n is *not* the diffracted wave. This has yet to be obtained from ψ_n.

In the general discussion of the diffraction effects at the open end, it was predicted that any wave, say $\psi_n^{(2)}$, emanating from the lower lip, would propagate unchanged in the region $-\frac{1}{2}\pi<\theta<\pi$, and would be totally reflected inside the channel $(-\pi<\theta<-\frac{1}{2}\pi)$. It follows that $\psi_{n+1}^{(1)}$ must contain the reflexion of $\psi_n^{(2)}$ for $-\pi<\theta<-\frac{1}{2}\pi$, and cancel $\psi_n^{(2)}$ for $\frac{1}{2}\pi<\theta<\pi$, in addition to supplying the $(n+1)$th diffracted wave $F_{n+1}^{(1)}$. To deduce $F_{n+1}^{(1)}$ from $\psi_{n+1}^{(1)}$, an expression for the total reflexion of $\psi_n^{(2)}$ at the upper wall is required. The most suitable form of this expression is obtained with the aid of the Green function for a doubly infinite plane, namely,

$$G'(r,\theta;r_0,\theta_0)=K_0[p(r^2+r_0^2-2rr_0\cos(\theta-\theta_0))^{\frac{1}{2}}]+K_0[p(r^2+r_0^2-2rr_0\cos(\theta+\theta_0))^{\frac{1}{2}}]. \quad (46)$$

It is clear that this expression is a solution of the wave equation having the correct singularity at the point (r_0,θ_0) and such that its normal derivative is zero along the plane $(\theta=0,\pi)$. Strictly, the above Green function relates to one of the regions either above the plane, or below, so that only one of its singularities, (r_0,θ_0), $(r_0,-\theta_0)$ lies in the region considered. It is for this reason that the above expression does not hold for the case of a semi-infinite plane where it is necessary to have only one singularity in the whole region $-\pi<\theta<\pi$.

In the application of this Green function, the contour of integration will be the whole of the plane $(\theta=0,\pi)$, together with a large semicircle whose radius tends to infinity.

Thus the image of $\psi_n^{(2)}$ in the upper wall of the channel can be written

$$[\psi_n^{(2)}]_{\text{ref.}}=-\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=0}K_0[p(r^2+\xi^2-2\xi r\cos\theta)^{\frac{1}{2}}]\,d\xi$$
$$+\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi}K_0[p(r^2+\xi^2+2\xi r\cos\theta)^{\frac{1}{2}}]\,d\xi. \quad (47)$$

Equation (44) can now be written

$$\left.\begin{aligned}\psi_{n+1}^{(1)}&=[\psi_n^{(2)}]_{\text{ref.}}+F_{n+1}^{(1)} &&(-\pi<\theta<-\tfrac{1}{2}\pi)\\ &=F_{n+1}^{(1)} &&(-\tfrac{1}{2}\pi<\theta<0),\end{aligned}\right\} \quad (48)$$

where, with the help of (43), (44) and (47),

$$\left.\begin{aligned}F_{n+1}^{(1)}&=\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=0}K_0[p(r^2+\xi^2-2\xi r\cos\theta)^{\frac{1}{2}}]\,d\xi\\ &\quad-\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi}d\xi\int_{-\infty}^{2(r\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}\frac{\exp[-p(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}]\,d\tau}{(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}}\\ &\qquad\qquad(-\pi<\theta<-\tfrac{1}{2}\pi)\\ &=\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi}K_0[p(r^2+\xi^2+2\xi r\cos\theta)^{\frac{1}{2}}]\,d\xi\\ &\quad-\frac{1}{\pi}\int_0^\infty\left\{\frac{\partial}{\xi\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi}d\xi\int_{-\infty}^{2(r\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}\frac{\exp[-p(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}]\,d\tau}{(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}}\\ &\qquad\qquad(-\tfrac{1}{2}\pi<\theta<0).\end{aligned}\right\} \quad (49)$$

It also follows, from equation (44), that $\psi_{n+1}^{(1)}$ is an odd function of θ, and therefore contains a term which cancels $\psi_n^{(2)}$ in the region $\frac{1}{2}\pi<\theta<\pi$. The remaining term, $F_{n+1}^{(1)}$, is such that

$$F_{n+1}^{(1)}(\theta)=-F_{n+1}^{(1)}(-\theta). \quad (50)$$

Equations (49) and (50) define $F^{(1)}_{n+1}$ in the whole range $-\pi<\theta<\pi$. These formulae, together with (44), constitute the basic results of the analysis. Equation (44) shows that each term in the expression for the potential can be deduced from the previous term, while (49) and (50) define the successive diffracted waves according to the original conception of the diffraction phenomena at the open end. Together with F_1, they represent a formal, but complete, solution to problem I. An analogous solution to problem II is no more difficult to obtain, but the interpretation in either case would be difficult and has not been attempted here. The asymptotic behaviour can, however, be deduced, and explicit interpretation for the first few values of n is practicable. This will be the object of the rest of the paper.

Asymptotic behaviour

The results of the preceding section enable the contribution to $f(\theta)$ from the $(n+1)$th diffracted waves, say $f_{n+1}(\theta)$, to be deduced, and hence, by the reciprocity relation, the corresponding terms occurring in the expressions for the absorbed waves, namely, C_{n+1}, B_{n+1}.

With the help of equation (49) and the asymptotic expression for the Bessel function,

$$K_0(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z}, \tag{51}$$

it can be shown that

$$\left.\begin{aligned} F^{(1)}_{n+1}(r,\theta) &\sim \frac{\mathrm{e}^{-pr}}{(2pr\pi)^{\frac{1}{2}}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=0} \exp\left[p\xi\cos\theta\right]d\xi \\ &\qquad -\frac{\mathrm{e}^{-pr}}{\pi}\left(\frac{2}{r}\right)^{\frac{1}{2}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta\right]d\xi\int_{-\infty}^{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \mathrm{e}^{-pu^2}\,du \\ &\qquad\qquad (-\pi<\theta<-\tfrac{1}{2}\pi) \\ &\sim \frac{\mathrm{e}^{-pr}}{(2pr\pi)^{\frac{1}{2}}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta\right]d\xi \\ &\qquad -\frac{\mathrm{e}^{-pr}}{\pi}\left(\frac{2}{r}\right)^{\frac{1}{2}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta\right]d\xi\int_{-\infty}^{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \mathrm{e}^{-pu^2}\,du \\ &\qquad\qquad (-\tfrac{1}{2}\pi<\theta<0), \end{aligned}\right\} \tag{52}$$

while, in the range $0<\theta<\pi$, $F^{(1)}_{n+1}$ is defined by (50).

As in the case of the first diffracted waves, the contribution to the potential outside the channel from the $(n+1)$th diffracted waves is given by

$$\left.\begin{aligned} &F^{(1)}_{n+1}+F^{(2)}_{n+1} \quad (0<\theta<\tfrac{1}{2}\pi), \\ &F^{(1)}_{n+1} \quad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{53}$$

The wave $F^{(2)}_{n+1}$ is, of course, similar in form to $F^{(1)}_{n+1}$ apart from a change of origin so that, from (50) and (52), the asymptotic behaviour outside the channel can be written

$$\left.\begin{aligned} F^{(1)}_{n+1}+F^{(2)}_{n+1} &\sim -(1-\exp\left[-2pb\sin\theta\right])\frac{\mathrm{e}^{-pr}}{r^{-\frac{1}{2}}}\Big[(2p\pi)^{-\frac{1}{2}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta\right]d\xi \\ &\qquad -\frac{2^{\frac{1}{2}}}{\pi}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta\right]d\xi\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} \mathrm{e}^{-pu^2}\,du\Big] \quad (0<\theta<\tfrac{1}{2}\pi) \\ F^{(1)}_{n+1} &\sim -\frac{\mathrm{e}^{-pr}}{r^{\frac{1}{2}}}\Big[(2p\pi)^{-\frac{1}{2}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=0} \exp\left[p\xi\cos\theta\right]d\xi \\ &\qquad -\frac{2^{\frac{1}{2}}}{\pi}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi^{(2)}_n(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta\right]d\xi\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} \mathrm{e}^{-pu^2}\,du\Big] \quad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{54}$$

Since the asymptotic form of the potential is denoted by $f(\theta)\,\mathrm{e}^{-pr}/r^{\frac{1}{2}}$ (equation (19)), the relation for $f_{n+1}(\theta)$ follows immediately. It is simply the coefficient of $\mathrm{e}^{-pr}/r^{\frac{1}{2}}$ in the above expression.

Equation (25) then gives, for the contribution to the potential inside the channel,

$$\left.\begin{aligned} C_{n+1} &= -\frac{1}{2pb}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta_0\right] d\xi \\ &\qquad + \frac{1}{b(p\pi)^{\frac{1}{2}}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta_0\right] d\xi \int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0}^\infty \mathrm{e}^{-pu^2}\,du \\ &\qquad\qquad (0<\theta_0<\tfrac{1}{2}\pi) \\ &= -\frac{1}{2pb}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=0} \exp\left[p\xi\cos\theta_0\right] d\xi \\ &\qquad + \frac{1}{b(p\pi)^{\frac{1}{2}}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi} \exp\left[-p\xi\cos\theta_0\right] d\xi \int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0}^\infty \mathrm{e}^{-pu^2}\,du \\ &\qquad\qquad (\tfrac{1}{2}\pi<\theta_0<\pi), \end{aligned}\right\} \tag{55}$$

where, as before, the factor $(1-\exp[-2pb\sin\theta_0])$, occurring for $0<\theta_0<\frac{1}{2}\pi$, is responsible for the two terms

$$C_{n+1}(t-x,\theta_0)\,H(t-x-2nb)-C_{n+1}(t-x-2b\sin\theta_0,\theta_0)\,H(t-x-2b\sin\theta_0-2nb)$$

(see equation (2)).

Finally, from (26) and (27),

$$\begin{aligned} 2C_{n+1}(\pi) = -B_{n+1} &= -\frac{1}{2pb}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=0} \mathrm{e}^{-p\xi}\,d\xi \\ &\qquad + \frac{1}{b(p\pi)^{\frac{1}{2}}}\int_0^\infty \left\{\frac{\partial}{\xi\,\partial\theta}\psi_n^{(2)}(\xi,\theta)\right\}_{\theta=\pi} \mathrm{e}^{p\xi}\,d\xi \int_{(2\xi)^{\frac{1}{2}}}^\infty \mathrm{e}^{-pu^2}\,du, \end{aligned} \tag{56}$$

which gives the contribution to the returning wave in problem I.

Equations (55) and (56) represent, physically, the most interesting results, since when interpreted they indicate the proportion of energy returning along the channel in the various intervals. It is to be noted that, in their evaluation, the bracketed term in the integrand refers to the normal derivative of the *total* correction term $\psi_n^{(2)}$, where $\psi_1^{(2)}=F_1^{(2)}$ and, for $n\geq 2$, $\psi_n^{(2)}$ is deduced from $\psi_{n-1}^{(2)}$ with the help of equation (44).

It may be remarked that these results can also be obtained directly from the image series. The correction terms, however, are rather involved and no further information was obtained from them.

Contribution from the second diffracted wave

The general formula of the preceding section can now be used to find the effect of the second diffracted waves. In this case $\psi_1^{(2)}$ reduces to $F_1^{(2)}$, since this is the only wave propagating from the lower lip during the interval $0\leq t\leq 2b$.

The expression for $F_1^{(2)}$ is readily deduced from equation (28) with a suitable change of origin,

$$F_1^{(2)} = -F\left(\frac{p}{\pi}\right)^{\frac{1}{2}} \exp\left[-pr\cos\theta\right]\int_{\{(r^2+4b^2+4br\sin\theta)^{\frac{1}{2}}-r\cos\theta\}^{\frac{1}{2}}}^\infty \mathrm{e}^{-p\lambda^2}\,d\lambda. \tag{57}$$

This holds for all points lying above the lower wall, which is sufficient for the present application.

It follows that

$$\left\{\frac{\partial}{\xi\,\partial\theta}F_1^{(2)}(\xi,\theta)\right\}_{\theta=0,\,\pi}=\pm bF\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}\right]}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}\mp\xi\}^{\frac{1}{2}}},\tag{58}$$

the upper or lower signs being taken according as the left-hand side is evaluated at $\theta=0$ or π.

This is now substituted in equation (55) to give

$$\left.\begin{aligned}C_2&=\frac{F}{2p^{\frac{1}{2}}\pi^{\frac{1}{2}}}\int_0^\infty\frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta_0\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}-\frac{F}{\pi}\int_0^\infty\frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta_0\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0}^\infty e^{-pu^2}\,du\\&\qquad(0<\theta_0<\tfrac{1}{2}\pi)\\&=\frac{-F}{2p^{\frac{1}{2}}\pi^{\frac{1}{2}}}\int_0^\infty\frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}+p\xi\cos\theta_0\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}-\xi\}^{\frac{1}{2}}}-\frac{F}{\pi}\int_0^\infty\frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta_0\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0}^\infty e^{-pu^2}\,du\\&\qquad(\tfrac{1}{2}\pi<\theta_0<\pi).\end{aligned}\right\}\tag{59}$$

The interpretation of this expression is carried out in appendix 1, where is is shown that

$$\left.\begin{aligned}C_2(t-x,\theta)&=\frac{1}{4b\pi\sin\frac{1}{2}\theta}\int_{2b}^{(t-x)}\mathfrak{F}(t-x-\tau)\cos^{-1}\left[\frac{\tau^2-8b^2\sin^2\frac{1}{2}\theta}{\tau(\tau^2-4b^2\sin^2\theta)^{\frac{1}{2}}}\right]d\tau\quad(0<\theta<\tfrac{1}{2}\pi)\\&=-\frac{1}{4b\pi\sin\frac{1}{2}\theta}\int_{2b}^{(t-x)}\mathfrak{F}(t-x-\tau)\cos^{-1}\left[\frac{8b^2\sin^2\frac{1}{2}\theta-\tau^2}{\tau(\tau^2-4b^2\sin^2\theta)^{\frac{1}{2}}}\right]d\tau\quad(\tfrac{1}{2}\pi<\theta<\pi),\end{aligned}\right\}\tag{60}$$

the inverse cosine to be taken between 0 and π.

$B_2(t-x)$ is immediately deduced from (60) by the substitution $\theta=\pi$ and a change of sign:

$$\begin{aligned}B_2(t-x)&=\frac{1}{4b\pi}\int_{2b}^{(t-x)}\mathfrak{F}(t-x-\tau)\cos^{-1}\left[\frac{8b^2-\tau^2}{\tau^2}\right]d\tau\\&=\frac{1}{2b\pi}\int_{2b}^{(t-x)}\mathfrak{F}(t-x-\tau)\cos^{-1}\left(\frac{2b}{\tau}\right)d\tau.\end{aligned}\tag{61}$$

Contribution from the third diffracted wave

Here the analysis is confined to the effects of a wave originating inside the channel. Even in this, the simplest case, the algebra involved is tedious and has been relegated to appendix 2, where it is shown that

$$B_3(t-x)=\frac{1}{2b}\int_{4b}^{(t-x)}\mathfrak{F}(t-x-\tau)\,h(\tau/2b)\,d\tau,\tag{62}$$

where
$$\begin{aligned}h(z)&=\frac{1}{\pi}\sin^{-1}\left(\frac{1}{z^{\frac{1}{2}}}\right)-\frac{1}{\pi}\sin^{-1}\left[\frac{z-(z^2-4)^{\frac{1}{2}}}{2z}\right]^{\frac{1}{2}}\\&\quad-\frac{1}{\pi^2}\int_1^{z-1}\tan^{-1}\left[\frac{zx-1}{x^2-1}\{(z-x)^2-1\}\right]^{\frac{1}{2}}\frac{dx}{x(zx-1)^{\frac{1}{2}}}.\end{aligned}\tag{63}$$

Unfortunately, the integral does not seem to be expressible in terms of known functions. In table 1, $h(z)$ is tabulated for values of z between 2 and 4.

Table 1. Values of $h(z)$ for $2 \leqslant z \leqslant 4$

z	$h(z)$	z	$h(z)$
2·0	0	3·0	0·03566
2·1	0·03433	3·1	0·03435
2·2	0·04009	3·2	0·03308
2·3	0·04197	3·3	0·03185
2·4	0·04224	3·4	0·03067
2·5	0·04171	3·5	0·02955
2·6	0·04076	3·6	0·02847
2·7	0·03959	3·7	0·02744
2·8	0·03831	3·8	0·02646
2·9	0·03699	3·9	0·02553
		4·0	0·02465

Numerical results and scope of application

In table 2 and figure 11 (p. 547) are given the results describing the behaviour of the returning wave due to a pulse of the form $\mathfrak{F}(t+x) = -(t+x)\,H(t+x)$ originating inside the channel. The particle velocity in the incident pulse is therefore represented by the Heaviside unit function and jumps discontinuously from zero to unity upon the arrival of the pulse at any given point of the channel. The calculations were carried out in terms of the velocity, since this is of more physical significance. In the returning wave this will be

$$\frac{\partial}{\partial x}\mathfrak{B}(t-x) = -\mathfrak{B}'(t-x), \tag{64}$$

where, from equation (1),

$$\mathfrak{B}' = B_1'(t-x)\,H(t-x) + B_2'(t-x)\,H(t-x-2b) + B_3'(t-x)\,H(t-x-4b), \tag{65}$$

and B_1', B_2', B_3' are derived from equations (35), (61) and (62) respectively, using the above form of $\mathfrak{F}(t+x)$. The resulting expressions are

$$B_1' = \frac{1}{4b}(t-x), \tag{66}$$

$$B_2' = -\frac{1}{2b\pi}\int_{2b}^{(t-x)} \cos^{-1}\left(\frac{2b}{\tau}\right) d\tau$$

$$= -\frac{1}{\pi}\left[\frac{t-x}{2b}\cos^{-1}\left(\frac{2b}{t-x}\right) - \cosh^{-1}\left(\frac{t-x}{2b}\right)\right], \tag{67}$$

$$B_3' = -\frac{1}{2b}\int_{4b}^{(t-x)} h\left(\frac{\tau}{2b}\right) d\tau. \tag{68}$$

The function B_3' was obtained by numerical integration from the values of $h(z)$ given in table 1.

The asymptotic value of $\mathfrak{B}'$ is unity, since the original pulse tends to behave like a harmonic wave of infinitely large wave-length. Such a wave is completely reflected at the open end with a change in sign.

The calculations are strictly valid for $0 \leqslant (t-x) \leqslant 6b$. Since, however, the contribution of B_3' is small for small values of the argument, the calculations are extended to include the region $6b \leqslant (t-x) \leqslant 8b$ assuming that the contribution from B_4' is negligible in this region. It will be seen that in the first region at the head of the returning wave, where the term B_1'

alone is evident, the velocity increases linearly from zero at the wave-front to one-half its maximum value at a distance $2b$ from the wave-front. Beyond this point B_2' becomes evident and, subsequently, B_3'; the increase in velocity then becomes more gradual until, at $(t-x) = 8b$, the velocity is within 8·9 % of its asymptotic value. Beyond this point, it will presumably tend steadily to its asymptotic value of unity.

It is interesting to note that the discontinuity at the head of the original pulse is smoothed out, and since any incident wave can be constructed by the superposition of elementary

Table 2. Contributions to the returning wave originating from a unit pulse

$(t-x)/20$	B_1'	B_2'	B_3'	$\mathfrak{B}'$
0	0	—	—	0
0·2	0·1	—	—	0·1
0·4	0·2	—	—	0·2
0·6	0·3	—	—	0·3
0·8	0·4	—	—	0·4
1·0	0·5	0	—	0·5
1·2	0·6	−0·0256	—	0·5744
1·4	0·7	0·0695	—	0·6305
1·6	0·8	0·1229	—	0·6771
1·8	0·9	0·1828	—	0·7172
2·0	1·0	$0·2474_5$	0	$0·7525_5$
2·2	1·1	0·3158	−0·0060	0·7782
2·4	1·2	0·3872	0·0143	0·7985
2·6	1·3	0·4609	0·0226	0·8165
2·8	1·4	0·5368	0·0306	0·8326
3·0	1·5	0·6144	0·0380	0·8476
3·2	1·6	0·6935	0·0448	0·8617
3·4	1·7	0·7739	0·0512	0·8749
3·6	1·8	0·8554	0·0571	0·8875
3·8	1·9	0·9380	0·0626	0·8994
4·0	2·0	$1·0214_5$	0·0677	$0·9108_5$

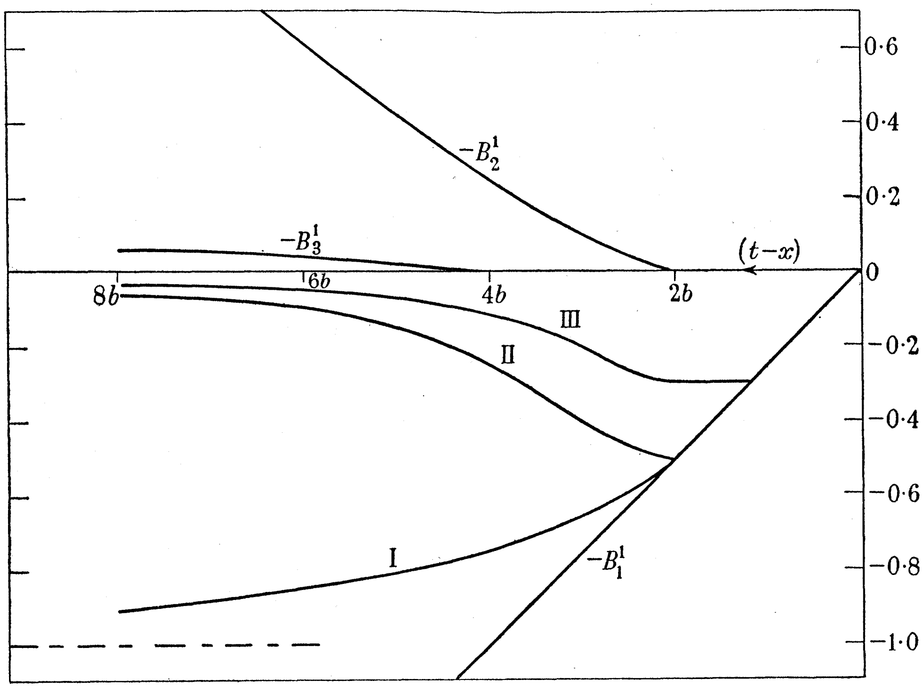

Figure 11. Velocity distribution in returning wave originating from: I, unit pulse, $H(t+x)$; II, finite pulse, $\{H(t+x)-H(t+x-2b)\}$; III, finite pulse, $\{H(t+x)-H(t+x-0·6(26))\}$.

step-functions, it follows that any finite discontinuity in the incident wave will disappear in the returning wave. In fact, by virtue of the integrating property in the first region, they become discontinuities in the velocity derivatives. This is verified by the two graphs in figure 11, which describe the returning wave originating from a finite pulse of length z in which the velocities are represented, respectively, by

$$-\{\mathfrak{B}'(t+x)-\mathfrak{B}'(t+x-2bz)\} \quad \text{and} \quad -\{H(t+x)-H(t+x-2bz)\},$$

where $\mathfrak{B}'$ is given by (65). The two cases $z = 0{\cdot}6$, $z = 1{\cdot}0$ are shown in the figure. The graph is everywhere continuous, but there is a discontinuity in slope at a distance z from the wave-front, where the velocity has a maximum value. Since the returning energy must remain finite the velocity in this case must tend ultimately to zero. For $z = 1$ and $(t-x) = 8b$ it is within 12·6 % of its maximum value.

While it is true that, from a practical point of view, the results obtained are capable of interpretation only within a certain length of the returning wave, and that any attempt to extend the calculations would involve an excessive amount of algebra, nevertheless, for an incident pulse of a step-function or truly transient type, the interesting characteristics of the returning wave are calculable, in the sense that the transition from zero disturbance at the wave-front, to a steady disturbance at large distances from the wave-front, takes place in the regions considered. If, however, the incident wave is of a periodic character, then the problem resolves itself into two parts. If there is a well-defined wave-front, the present analysis will still give the behaviour at the head of the returning wave, but at large distances from the wave-front the disturbance will again be periodic and the solution is no longer trivial. This latter problem is solved by separating the incident wave into its harmonic spectrum and suppressing the time factor. The analysis involved is quite different, and it is hoped to discuss this in a later paper.

Reflected and transmitted energy

When the returning wave has become sensibly plane, so that the potential can be represented by $\mathfrak{B}(t-x)$, the energy in length s of the wave is

$$\tfrac{1}{2}\rho(2b)\int_0^s \left\{\frac{\partial \mathfrak{B}(u)}{\partial u}\right\}^2 du, \tag{69}$$

where s is measured from the wave-front and ρ is the undisturbed density.

Thus, in the first region of length $2b$, where the only contribution to $\mathfrak{B}$ is given by B_1, the energy, compared with that in length $2b$ at the head of the original wave, will be

$$\int_0^{2b} \{\mathfrak{F}(u)\}^2\, du \Big/ 16b^2 \int_0^{2b} \left\{\frac{\partial \mathfrak{F}(u)}{\partial u}\right\}^2 du. \tag{70}$$

For an incident pulse $\mathfrak{F}(t+x) = -(t+x)\,H(t+x)$ it is easy to see that this ratio is $1/12$.

It is, perhaps, of interest to note that the function which makes (70) a maximum, and such that $\mathfrak{F}(0) = 0$, is known to be $\mathfrak{F}(u) = \sin(\pi u/4b)$ (Hardy, Littlewood & Polya 1934, p. 184). The energy ratio in this case is $1/\pi^2$.

Outside the channel, the regions affected only by the first diffracted wave $F_1^{(1)}$ are those lying between $(t-2b)<r<t$ $(\frac{1}{2}\pi<\theta<\pi)$ and $r<t$, $r_2>t$ $(\theta_c<\theta<\frac{1}{2}\pi)$, where r_2 is measured from the lower lip and θ_c is the value of θ at the intersection of the two circles $r = t$, $r_2 = t$.

In addition, there is a region $(t-2b)<r<t$, $(t-2b)<r_2<t$ $(-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi)$, in which only $F_1^{(1)}$ and $F_1^{(2)}$ occur (see figure 5). The energy which is propagated to infinity within these regions can therefore be calculated from the asymptotic relations for the first diffracted waves.

In the region $\frac{1}{2}\pi<\theta<\pi$, for example, the asymptotic form of $F_1^{(1)}$ for large r ($(t-r)$ remaining finite) is readily deduced from equation (14):

$$F_1^{(1)} \sim \frac{1}{(2r)^{\frac{1}{2}}\pi\sin\frac{1}{2}\theta}\int_0^{(t-r)^{\frac{1}{2}}} F(t-r-z^2)\,dz. \tag{71}$$

The velocity, q, is given by

$$q^2 = \left(\frac{\partial F_1^{(1)}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F_1^{(1)}}{\partial \theta}\right)^2 \sim \frac{1}{2r\pi^2\sin^2\frac{1}{2}\theta}\left[\int_0^{(t-r)^{\frac{1}{2}}} \mathfrak{F}'(t-r-z^2)\,dz\right]^2, \tag{72}$$

and, in the limit as $r\to\infty$, the energy is

$$E_1 = \tfrac{1}{2}\rho\frac{1}{2\pi^2}\int_{\frac{1}{2}\pi}^{\pi} d\theta \int_{(t-2b)}^{t} \frac{1}{\sin^2\frac{1}{2}\theta}\left[\int_0^{(t-r)^{\frac{1}{2}}} \mathfrak{F}'(t-r-z^2)\,dz\right]^2 dr$$
$$= \frac{\rho}{2\pi^2}\int_0^{2b}\left[\int_0^{u^{\frac{1}{2}}} \mathfrak{F}'(u-y^2)\,dy\right]^2 du. \tag{73}$$

If the original wave is a unit pulse, so that $\mathfrak{F}'(x)=H(x)$, then

$$E_1 = \rho b^2/\pi^2. \tag{74}$$

For the region $0<\theta<\frac{1}{2}\pi$, $r<t$, $r_2>t$, the asymptotic value of r_2 is $(r+2b\sin\theta)$, so that the energy is

$$E_2 = \lim_{\substack{r\to\infty\\(t-r)\text{ finite}}} \tfrac{1}{2}\rho\int_0^{\frac{1}{2}\pi} d\theta \int_{t-2b\sin\theta}^{t}\left[\left(\frac{\partial F_1^{(1)}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F_1^{(1)}}{\partial\theta}\right)^2\right] dr$$
$$= \lim_{\delta\to 0}\lim_{r\to\infty} \tfrac{1}{2}\rho\int_\delta^{\frac{1}{2}\pi} d\theta \int_{t-2b\sin\theta}^{t}\left[\left(\frac{\partial F_1^{(1)}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F_1^{(1)}}{\partial\theta}\right)^2\right] dr, \tag{75}$$

since the energy contained in the region $\delta<\theta<\frac{1}{2}\pi$ must be a continuous function of δ. In this region the asymptotic form of $F_1^{(1)}$ may be used, and (75) then reduces to

$$E_2 = \tfrac{1}{2}\rho\frac{1}{2\pi^2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sin^2\frac{1}{2}\theta}\int_0^{2b\sin\theta}\left[\int_0^{u^{\frac{1}{2}}} \mathfrak{F}'(u-z^2)\,dz\right]^2 du$$
$$= \frac{\rho}{2\pi^2}\int_0^{2b}\left[\int_0^{u^{\frac{1}{2}}} \mathfrak{F}'(u-z^2)\,dz\right]^2\left[\frac{2b+(4b^2-u^2)^{\frac{1}{2}}}{u}-1\right] du. \tag{76}$$

For a unit pulse, this becomes

$$E_2 = \rho b^2\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right). \tag{77}$$

The potential of the disturbance lying within the extensions of the channel walls tends to zero at infinity, since the contributions from $F_1^{(1)}$ and $F_1^{(2)}$ tend to annul the incident pulse which continues to propagate in this region.

The only region still to consider is, therefore, effectively

$$(t-2b)<r<t, \quad (t-2b)<r_2<t \quad (0<\theta<\tfrac{1}{2}\pi),$$

where both the first diffracted waves occur and

$$F_1^{(1)}+F_1^{(2)} \sim \frac{1}{(2r)^{\frac{1}{2}}\pi\sin\frac{1}{2}\theta}\left[\int_0^{(t-r)^{\frac{1}{2}}}\mathfrak{F}(t-r-z^2)\,dz-\int_0^{(t-r-2b\sin\theta)^{\frac{1}{2}}}\mathfrak{F}(t-r-2b\sin\theta-z^2)\,dz\right]. \quad (78)$$

The energy is, therefore, in the limit

$$E_3 = \frac{\rho}{4\pi^2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sin^2\frac{1}{2}\theta}\int_{t-2b}^{t-2b\sin\theta}\left[\int_0^{(t-r)^{\frac{1}{2}}}\mathfrak{F}'(t-r-z^2)\,dz-\int_0^{(t-r-2b\sin\theta)^{\frac{1}{2}}}\mathfrak{F}'(t-r-2b\sin\theta-z^2)dz\right]^2 dr$$

$$= \frac{\rho}{4\pi^2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sin^2\frac{1}{2}\theta}\int_{2b\sin\theta}^{2b}\left[\int_0^{u^{\frac{1}{2}}}\mathfrak{F}'(u-z^2)\,dz-\int_0^{(u-2b\sin\theta)^{\frac{1}{2}}}\mathfrak{F}'(u-2b\sin\theta-z^2)\,dz\right]^2 du, \quad (79)$$

and for a unit pulse

$$E_3 = \frac{\rho}{4\pi^2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sin^2\frac{1}{2}\theta}\int_{2b\sin\theta}^{2b}[2u-2b\sin\theta-2u^{\frac{1}{2}}(u-2b\sin\theta)^{\frac{1}{2}}]\,du$$

$$= \frac{\rho b^2}{4\pi^2}\int_0^{\frac{1}{2}\pi}[4(1-\sin\theta)-2(2-\sin\theta)\,(1-\sin\theta)^{\frac{1}{2}}+2\sin^2\theta\log\{1+(1-\sin\theta)^{\frac{1}{2}}\}$$
$$-\sin^2\theta\log\sin\theta]\frac{d\theta}{\sin^2\frac{1}{2}\theta} \quad (80)$$

$$= \frac{\rho b^2 I}{4\pi^2} \quad \text{(say)}, \quad (81)$$

where, after some reduction,

$$I = 16\log(4-2^{\frac{3}{2}})-4(3-2^{\frac{3}{2}})+2\pi\log 2+24\int_0^{\frac{1}{4}\pi}\log\sin\theta\,d\theta-32\int_0^{\frac{1}{8}\pi}\log\sin\theta\,d\theta$$
$$= 8{\cdot}4671. \quad (82)$$

Since there is an equivalent amount of energy in the corresponding regions of the lower half-plane, it follows from (74), (77) and (81) that the total energy transmitted to infinity, in the regions containing only the first diffracted waves, is

$$2(E_1+E_2+E_3) = 2\rho b^2\left\{\frac{1}{\pi^2}+\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)+\frac{I}{4\pi^2}\right\} = 0{\cdot}5763(2\rho b^2), \quad (83)$$

or 57·6 % of the original energy contained in the length $2b$ at the head of the incident pulse is transmitted in these regions. In the case of the finite pulse, namely

$$\mathfrak{F} = -\{(t+x)\,H(t+x)-(t+x-2b)\,H(t+x-2b)\}$$

this, of course, represents the percentage of total energy.

No further calculations were attempted for the transmitted wave, but the energy in the returning wave was evaluated as far as $(t-x) = 8b$. Two cases were considered, the Heaviside unit pulse for which $\mathfrak{F} = -(t+x)\,H(t+x)$, and the finite pulse

$$\mathfrak{F} = -\{(t+x)\,H(t+x)-(t+x-2b)\,H(t+x-2b)\}.$$

The results of the calculations are exhibited in figure 12.

In the first case the function $E_h(s)/\frac{1}{2}\rho(2bs)$ is plotted, where $E_h(s)$ is the energy in length s at the head of the returning wave and $\frac{1}{2}\rho(2bs)$ that in the corresponding length of the incident pulse. The asymptotic value is unity if the pulse continues to propagate indefinitely, although the results are valid, as far as they go, if the pulse is of any finite length greater than $8b$. It is, in fact, implicit in the calculations that the pulse will terminate at some point, since it has been tacitly assumed that the region in which the energy is calculated is affected only by the returning wave.

In the second case $E_f(s)/\frac{1}{2}\rho(2b)^2$ is plotted giving the proportion of total energy returning along the channel. The asymptotic value of this quantity is not known, although, from (83), it is certainly less than 0·43. A study of figure 12 suggests that it is actually much lower than this and is probably in the region of 0·28.

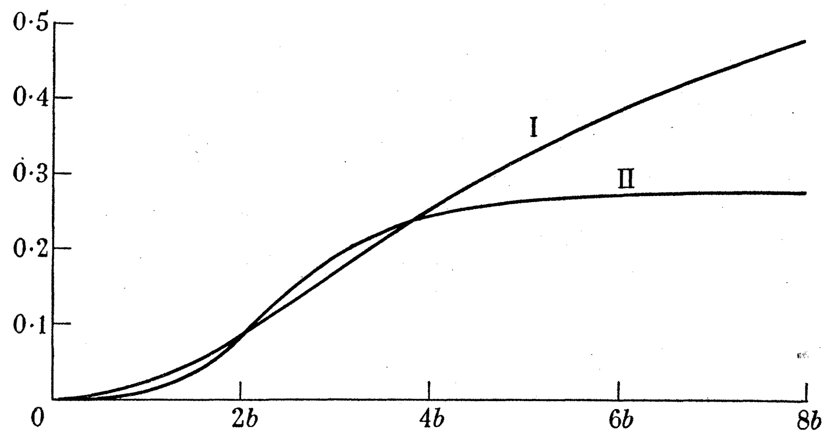

FIGURE 12. I, $\frac{E_h(s)}{\frac{1}{2}\rho(2bs)}$, reflected energy originating from unit pulse;
II, $\frac{E_f(s)}{\frac{1}{2}\rho(2b)^2}$, reflected energy originating from finite pulse.

In conclusion, I should like to thank Professor Goldstein and Mr Lighthill for their helpful criticism during the preparation of this paper.

APPENDIX 1

The interpretation of

$$\left.\begin{aligned} C_2 &= \frac{F}{2(p\pi)^{\frac{1}{2}}}\int_0^\infty \frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}-\frac{F}{\pi}\int_0^\infty \frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^\infty e^{-pu^2}\,du \\ &\qquad (0<\theta<\tfrac{1}{2}\pi) \\ &= -\frac{F}{2(p\pi)^{\frac{1}{2}}}\int_0^\infty \frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}+p\xi\cos\theta\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}-\xi\}^{\frac{1}{2}}}-\frac{F}{\pi}\int_0^\infty \frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^\infty e^{-pu^2}\,du \\ &\qquad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{A.1}$$

Since $$p^{\frac{1}{2}} = (\pi t)^{-\frac{1}{2}}H(t),$$
it follows that, for $0<\theta<\frac{1}{2}\pi$,

$$\begin{aligned} &\frac{1}{2}\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{\exp\left[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta\right]d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}} \\ &\quad = \frac{1}{2\pi}\int_0^{t=\xi\cos\theta+(\xi^2+4b^2)^{\frac{1}{2}}} \frac{d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}\{t-\xi\cos\theta-(\xi^2+4b^2)^{\frac{1}{2}}\}^{\frac{1}{2}}} \\ &\quad = \frac{1}{2\pi(2b)^{\frac{1}{2}}}\int_{z_1}^1 \frac{dz}{[tz-2b\sin^2\frac{1}{2}\theta z^2-2b\cos^2\frac{1}{2}\theta]^{\frac{1}{2}}}, \end{aligned} \tag{A.2}$$

where z_1 is the smaller root of the denominator. Similarly

$$p\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} e^{-pu^2}\,du = \frac{1}{2(t)^{\frac{1}{2}}} H(t-2\xi\sin^2\tfrac{1}{2}\theta), \tag{A. 3}$$

so that

$$\frac{p}{\pi}\int_0^\infty \frac{\exp[-p(\xi^2+4b^2)^{\frac{1}{2}}-p\xi\cos\theta]\,d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}\int_{(2\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} e^{-pu^2}\,du$$
$$= \frac{1}{2\pi}\int_0^{(t/2b=\sinh u\cos\theta+\cosh u+2\sinh u\sin^2\frac{1}{2}\theta)} \frac{e^{-\frac{1}{2}u}\,du}{\{t-2b\sinh u\cos\theta-2b\cosh u\}^{\frac{1}{2}}}$$
$$= \frac{1}{2\pi(2b)^{\frac{1}{2}}}\int_{2b/t}^{1}\frac{dz}{\{tz-2b\sin^2\frac{1}{2}\theta z^2-2b\cos^2\frac{1}{2}\theta\}^{\frac{1}{2}}}. \tag{A. 4}$$

The difference between (A. 2) and (A. 4) is then

$$\frac{1}{2\pi(2b)^{\frac{1}{2}}}\int_{z_1}^{2b/t}\frac{dz}{[tz-2bz^2\sin^2\frac{1}{2}\theta-2b\cos^2\frac{1}{2}\theta]^{\frac{1}{2}}} = \frac{1}{4b\pi\sin\frac{1}{2}\theta}\cos^{-1}\left[\frac{t^2-8b^2\sin^2\frac{1}{2}\theta}{t(t^2-4b^2\sin^2\theta)^{\frac{1}{2}}}\right]$$
$$(0<\theta<\tfrac{1}{2}\pi;\ t>2b), \tag{A. 5}$$

where the inverse cosine lies between 0 and $\frac{1}{2}\pi$.

When $\frac{1}{2}\pi<\theta<\pi$, the transform of the first term in C_2 becomes (omitting the factor F/p)

$$-\frac{1}{2\pi}\int_0^{t=(\xi^2+4b^2)^{\frac{1}{2}}-\xi\cos\theta}\frac{d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}-\xi\}^{\frac{1}{2}}\{t-(\xi^2+4b^2)^{\frac{1}{2}}+\xi\cos\theta\}^{\frac{1}{2}}}$$
$$= -\frac{1}{2\pi(2b)^{\frac{1}{2}}}\int_1^{z_2}\frac{dz}{[tz-2bz^2\sin^2\frac{1}{2}\theta-2b\cos^2\frac{1}{2}\theta]^{\frac{1}{2}}}, \tag{A. 6}$$

where z_2 is the larger root of the denominator.

The second term in C_2 remains unchanged, and the difference between (A. 6) and (A. 4) is

$$-\frac{1}{2\pi(2b)^{\frac{1}{2}}}\int_{2b/t}^{z_2}\frac{dz}{[tz-2bz^2\sin^2\frac{1}{2}\theta-2b\cos^2\frac{1}{2}\theta]^{\frac{1}{2}}}$$
$$= -\frac{1}{4b\pi\sin\frac{1}{2}\theta}\cos^{-1}\left[\frac{8b^2\sin^2\frac{1}{2}\theta-t^2}{t\{t^2-4b^2\sin^2\theta\}^{\frac{1}{2}}}\right] \quad (\tfrac{1}{2}\pi<\theta<\pi;\ t>2b), \tag{A. 7}$$

where the inverse cosine lies between 0 and π.

Finally, since $F(p) = \mathfrak{F}(t)$, it follows from the principle of superposition that

$$\left.\begin{aligned} C_2(t-x,\theta) &= \frac{1}{4b\pi\sin\frac{1}{2}\theta}\int_{2b}^{(t-x)}\mathfrak{F}(t-x-\tau)\cos^{-1}\left[\frac{\tau^2-8b^2\sin^2\frac{1}{2}\theta}{\tau(\tau^2-4b^2\sin^2\theta)^{\frac{1}{2}}}\right]d\tau \quad (0<\theta<\tfrac{1}{2}\pi) \\ &= -\frac{1}{4b\pi\sin\frac{1}{2}\theta}\int_{2b}^{(t-x)}\mathfrak{F}(t-x-\tau)\cos^{-1}\left[\frac{8b^2\sin^2\frac{1}{2}\theta-\tau^2}{\tau(\tau^2-4b^2\sin^2\theta)^{\frac{1}{2}}}\right]d\tau \quad (\tfrac{1}{2}\pi<\theta<\pi). \end{aligned}\right\} \tag{A. 8}$$

Appendix 2

For the interpretation of B_3, it is convenient to express $\psi_2^{(2)}$ as a function of Cartesian co-ordinates at the *lower* lip of the channel, say $\sigma_2(x,y)$. Equation (62) then becomes, for $n=2$,

$$B_3 = \frac{1}{2pb}\int_0^\infty\left\{\frac{\partial}{\partial y}\sigma_2(-\eta,y)\right\}_{y=2b} e^{-p\eta}\,d\eta + \frac{1}{b(p\pi)^{\frac{1}{2}}}\int_0^\infty\left\{\frac{\partial}{\partial y}\sigma_2(\eta,y)\right\}_{y=2b} e^{p\eta}\,d\eta\int_{(2\eta)^{\frac{1}{2}}}^{\infty} e^{-pv^2}\,dv, \tag{A. 9}$$

and σ_2 can be deduced immediately from $\psi_2^{(1)}$, with the help of equations (44) and (56). The explicit expression is

$$\sigma_2(x,y) = \sigma_2(-r\cos\theta, -r\sin\theta) = \frac{1}{\pi}\int_0^\infty g(\xi)\,d\xi \int_0^{2(r\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \frac{\exp[-p(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}]\,d\tau}{(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}}, \tag{A. 10}$$

where
$$g(\xi) = \frac{b(p/\pi)^{\frac{1}{2}} F \exp[-p(\xi^2+4b^2)^{\frac{1}{2}}]}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}+\xi\}^{\frac{1}{2}}}. \tag{A. 11}$$

Since, for $0<\theta<\pi$,

$$\int_0^{2(r\xi)^{\frac{1}{2}}\sin\frac{1}{2}\theta} \frac{\exp[-p(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}]\,d\tau}{(r^2+\xi^2+2\xi r\cos\theta+\tau^2)^{\frac{1}{2}}} = \int_{(r^2+\xi^2+2\xi r\cos\theta)^{\frac{1}{2}}}^{r+\xi} \frac{e^{-pu}\,du}{(u^2-r^2-\xi^2-2\xi r\cos\theta)^{\frac{1}{2}}}, \tag{A. 12}$$

and is an odd function of θ, it follows that, for $y>0$,

$$\sigma_2(x,y) = -\frac{1}{\pi}\int_0^\infty g(\xi)\,d\xi \int_{(x^2+y^2+\xi^2-2\xi x)^{\frac{1}{2}}}^{(x^2+y^2)^{\frac{1}{2}}+\xi} \frac{e^{-pu}\,du}{(u^2-x^2-y^2-\xi^2+2\xi x)^{\frac{1}{2}}}, \tag{A. 13}$$

and

$$B_3 = -\frac{\partial}{\partial y}\left[\frac{1}{2pb\pi}\int_0^\infty g(\xi)\,d\xi\int_0^\infty e^{-p\eta}\,d\eta \int_{(\eta^2+y^2+\xi^2+2\xi\eta)^{\frac{1}{2}}}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi} \frac{e^{-pu}\,du}{(u^2-\eta^2-y^2-\xi^2-2\xi\eta)^{\frac{1}{2}}}\right.$$
$$\left. +\frac{1}{b\pi(p\pi)^{\frac{1}{2}}}\int_0^\infty g(\xi)\,d\xi\int_0^\infty e^{p\eta}\,d\eta \int_{(\eta^2+y^2+\xi^2-2\xi\eta)^{\frac{1}{2}}}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi} \frac{e^{-pu}\,du}{(u^2-\eta^2-y^2-\xi^2+2\xi\eta)^{\frac{1}{2}}}\int_{(2\eta)^{\frac{1}{2}}}^\infty e^{-pv^2}\,dv\right]_{y=2b}. \tag{A. 14}$$

The differentiation with respect to y being taken outside the integration. This is justified since the integrals are uniformly convergent ($\sigma_2 \to 0$ as $x\to\infty$ for all y).

Consider first

$$\int_0^\infty e^{-p\eta}\,d\eta \int_{(\eta^2+y^2+\xi^2+2\xi\eta)^{\frac{1}{2}}}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi} \frac{e^{-pu}\,du}{(u^2-\eta^2-y^2-\xi^2-2\xi\eta)^{\frac{1}{2}}} = \int_0^\infty d\eta \int_{\eta+(\eta^2+y^2+\xi^2+2\xi\eta)^{\frac{1}{2}}}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi+\eta} \frac{e^{-pu}\,du}{\{u^2-y^2-\xi^2-2\eta\overline{u+\xi}\}^{\frac{1}{2}}}. \tag{A. 15}$$

Reversing the order of integration, and simplifying, we get

$$\int_{(y^2+\xi^2)^{\frac{1}{2}}}^{y+\xi} \frac{(u^2-y^2-\xi^2)^{\frac{1}{2}}\,e^{-pu}\,du}{u+\xi} + (2\xi)^{\frac{1}{2}}\,y\int_{y+\xi}^\infty \frac{e^{-pu}\,du}{(u+\xi)\,(u-\xi)^{\frac{1}{2}}}. \tag{A. 16}$$

Now consider

$$\int_0^\infty e^{p\eta}\,d\eta \int_{(\eta^2+y^2+\xi^2-2\xi\eta)^{\frac{1}{2}}}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi} \frac{e^{-pu}\,du}{(u^2-\eta^2-y^2-\xi^2+2\xi\eta)^{\frac{1}{2}}}\int_{(2\eta)^{\frac{1}{2}}}^\infty e^{-pv^2}\,dv \tag{A. 17}$$

$$= \int_0^\infty d\eta \int_{-\eta+(\eta^2+y^2+\xi^2-2\xi\eta)^{\frac{1}{2}}}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi-\eta} \frac{e^{-pu}\,du}{(u^2+2u\eta-y^2-\xi^2+2\xi\eta)^{\frac{1}{2}}}\int_{(2\eta)^{\frac{1}{2}}}^\infty e^{-pv^2}\,dv$$

$$= \left[\int_{(y^2+\xi^2)^{\frac{1}{2}}}^{y+\xi} du \int_0^{(y^2+2u\xi-u^2-\xi^2)/2(u-\xi)} d\eta + \int_\xi^{(y^2+\xi^2)^{\frac{1}{2}}} du \int_{(y^2+\xi^2-u^2)/2(u+\xi)}^{(y^2+2u\xi-u^2-\xi^2)/2(u-\xi)} d\eta\right]$$
$$\times\left[\frac{e^{-pu}}{\{u^2-y^2-\xi^2+2\eta(u+\xi)\}^{\frac{1}{2}}}\int_{(2\eta)^{\frac{1}{2}}}^\infty e^{-pv^2}\,dv\right]. \tag{A. 18}$$

Integrating by parts we obtain

$$(2\xi)^{\frac{1}{2}} y \int_{\xi}^{(y+\xi)} \frac{e^{-pu}\,du}{(u-\xi)^{\frac{1}{2}}(u+\xi)} \int_{(y^2+2u\xi-u^2-\xi^2)^{\frac{1}{2}}/(u-\xi)^{\frac{1}{2}}}^{\infty} e^{-pv^2}\,dv - \frac{1}{2}\left(\frac{\pi}{p}\right)^{\frac{1}{2}} \int_{(y^2+\xi^2)^{\frac{1}{2}}}^{y+\xi} \frac{e^{-pu}(u^2-y^2-\xi^2)^{\frac{1}{2}}\,du}{u+\xi}$$
$$+\int_0^{\infty} d\eta \int_{(\eta^2+y^2+\xi^2-2\xi\eta)^{\frac{1}{2}}-\eta}^{(\eta^2+y^2)^{\frac{1}{2}}+\xi-\eta} \frac{e^{-pu-2p\eta}(u^2-y^2-\xi^2+2\eta\overline{u+\xi})^{\frac{1}{2}}\,du}{(2\eta)^{\frac{1}{2}}(u+\xi)}. \quad \text{(A. 19)}$$

Substitution of (A. 16) and (A. 19) in (A. 14), and the use of the relations

$$\frac{1}{2}\left(\frac{\pi}{p}\right)^{\frac{1}{2}} = \int_0^{\infty} \frac{e^{-2p\eta}}{(2\eta)^{\frac{1}{2}}}\,d\eta, \quad \text{(A. 20)}$$

$$\left[\int_{y+\xi}^{\infty} du \int_0^{\infty} d\eta + \int_{\xi}^{\xi+y} du \int_{(y^2+2u\xi-u^2-\xi^2)/2(u-\xi)}^{\infty} d\eta\right] = \int_0^{\infty} d\eta \int_{\xi-\eta+(\eta^2+y^2)^{\frac{1}{2}}}^{\infty} du \quad \text{(A. 21)}$$

gives the following expression for B_3:

$$B_3 = -\frac{\partial}{\partial y}\int_0^{\infty} g(\xi)\,d\xi\left[\frac{\xi^{\frac{1}{2}}y}{b\pi(p\pi)^{\frac{1}{2}}}\int_0^{\infty}\frac{e^{-2p\eta}}{\eta^{\frac{1}{2}}}\,d\eta\int_{\xi-\eta+(\eta^2+y^2)^{\frac{1}{2}}}^{\infty}\frac{e^{-pu}\,du}{(u+\xi)(u-\xi)^{\frac{1}{2}}}\right.$$
$$\left.+\frac{1}{b\pi(p\pi)^{\frac{1}{2}}}\int_0^{\infty}\frac{e^{-2p\eta}}{(2\eta)^{\frac{1}{2}}}\,d\eta\int_{(y^2+\eta^2+\xi^2-2\xi\eta)^{\frac{1}{2}}-\eta}^{(y^2+\eta^2)^{\frac{1}{2}}+\xi-\eta}\frac{e^{-pu}\{u^2-y^2-\xi^2+2\eta(u+\xi)\}\,du}{u+\xi}\right]_{y=2b}. \quad \text{(A. 22)}$$

Performing the differentiation, and substituting for $g(\xi)$, we get

$$B_3 = -\frac{F}{p\pi^2}\int_0^{\infty}\frac{\exp[-p(\xi^2+4b^2)^{\frac{1}{2}}]\,d\xi}{(\xi^2+4b^2)^{\frac{1}{2}}\{(\xi^2+4b^2)^{\frac{1}{2}}-\xi\}^{\frac{1}{2}}}\int_0^{\infty}\frac{e^{-2p\eta}\,d\eta}{\eta^{\frac{1}{2}}}\left[p\xi^{\frac{1}{2}}\int_{\xi-\eta+(\eta^2+4b^2)^{\frac{1}{2}}}^{\infty}\frac{e^{-pu}\,du}{(u+\xi)(u-\xi)^{\frac{1}{2}}}\right.$$
$$\left.-2^{\frac{1}{2}}pb\int_{(\eta^2+\xi^2-2\xi\eta+4b^2)^{\frac{1}{2}}-\eta}^{(\eta^2+4b^2)^{\frac{1}{2}}+\xi-\eta}\frac{e^{-pu}\,du}{(u+\xi)\{u^2-4b^2-\xi^2+2\eta(u+\xi)\}^{\frac{1}{2}}}\right]. \quad \text{(A. 23)}$$

The interpretation of the bracketed expression is

$$\frac{H\{t-(\eta^2+\xi^2-2\xi\eta+4b^2)^{\frac{1}{2}}+\eta\}}{(t+\xi)\{t^2-\xi^2-4b^2+2t\eta+2\xi\eta\}^{\frac{1}{2}}}-\left[\frac{\xi^{\frac{1}{2}}}{(t+\xi)(t-\xi)^{\frac{1}{2}}}+\frac{2^{\frac{1}{2}}b}{(t+\xi)\{t^2-4b^2-\xi^2+2\eta(t+\xi)\}^{\frac{1}{2}}}\right]$$
$$\times H\{t+\eta-\xi-(\eta^2+4b^2)^{\frac{1}{2}}\}, \quad \text{(A. 24)}$$

and, combined with the 'η' integral, this yields

$$H\{t-(\xi^2+4b^2)^{\frac{1}{2}}\}\int_0^{t=\eta+(\eta^2+\xi^2-2\xi\eta+4b^2)^{\frac{1}{2}}}\frac{d\eta}{\eta^{\frac{1}{2}}(t+\xi-2\eta)\{t^2-\xi^2-4b^2-2\eta(t-\xi)\}^{\frac{1}{2}}}$$
$$-H(t-2b-\xi)\int_0^{t-\xi=\eta+(\eta^2+4b^2)^{\frac{1}{2}}}\left[\frac{\xi^{\frac{1}{2}}}{(t-\xi-2\eta)^{\frac{1}{2}}}+\frac{2^{\frac{1}{2}}b}{\{t^2-4b^2-\xi^2-2t\eta+2\xi\eta\}^{\frac{1}{2}}}\right]\frac{d\eta}{\eta^{\frac{1}{2}}(t+\xi-2\eta)}, \quad \text{(A. 25)}$$

$$=\frac{\pi H\{t-(\xi^2+4b^2)^{\frac{1}{2}}\}}{2^{\frac{3}{2}}b(t+\xi)^{\frac{1}{2}}}-\frac{H(t-\xi-2b)}{(t+\xi)^{\frac{1}{2}}}\tan^{-1}\left[\frac{\{(t-\xi)^2-4b^2\}\{t+\xi\}}{8b^2\xi}\right]^{\frac{1}{2}}. \quad \text{(A. 26)}$$

This is now combined with the 'ξ' integral of (A. 23), which is interpreted to give, after some reduction,

$$\frac{\pi H(t-4b)}{2b}\left[\sin^{-1}\left(\frac{2b}{t}\right)^{\frac{1}{2}}-\sin^{-1}\left\{\frac{t/2b-(t^2/4b^2-4)^{\frac{1}{2}}}{2t/2b}\right\}^{\frac{1}{2}}\right]$$
$$-\frac{H(t-4b)}{2b}\int_0^{t/2b-1}\tan^{-1}\left[\frac{tx/2b-1}{x^2-1}\{(t/2b-x)^2-1\}\right]^{\frac{1}{2}}\frac{dx}{x(tx/2b-1)^{\frac{1}{2}}}$$
$$=\frac{\pi^2H(t-4b)}{2b}\,h\left(\frac{t}{2b}\right)\quad\text{(say)}. \quad \text{(A. 27)}$$

Finally, since $F(p) = \mathfrak{F}(t)$, the interpretation of B_3 is obtained, by the principle of superposition, as

$$B_3(t-x) = \frac{1}{2b}\int_{4b}^{(t-x)} \mathfrak{F}(t-x-\tau)\, h\left(\frac{\tau}{2b}\right) d\tau, \tag{A. 28}$$

where h is defined by (A. 27).

Appendix 3

The application of the operational calculus to the present problem is briefly discussed below.

The relation between $\psi(t)\,H(t)$ and its operational representation $\Psi(p)$ is (Jeffreys & Jeffreys 1946)

$$\psi(t)\,H(t) = \frac{1}{2\pi i}\int_{y-i\infty}^{y+i\infty} \frac{e^{zt}}{z}\Psi(z)\,dz, \tag{A. 29}$$

where all the singularities of $\Psi(z)$ lie to the left of the contour of integration. It follows immediately from (A. 29) that

$$\psi(t+t_1)\,H(t+t_1) = \frac{1}{2\pi i}\int_{y-i\infty}^{y+i\infty} \frac{e^{z(t+t_1)}}{z}\Psi(z)\,dz, \tag{A. 30}$$

so that $\Psi(p)\,e^{pt_1}$ may be regarded as the operational representation of $\psi(t+t_1)\,H(t+t_1)$ provided that the contour of integration is chosen to lie to the right of the singularities of $\Psi(z)$.

In the present problem the potential consists of two parts; that due to the original wave which for simplicity will be assumed to originate inside the channel so that the potential is of the form $\mathfrak{F}(t+x)\,H(t+x)$ and is confined to the region $-2b \leqslant y \leqslant 0$ (figure 1). The remaining contribution from the diffracted waves, $\psi(x,y,t)$ (say), satisfies the condition $\psi = 0$, $t<0$ so that $\Psi(x,y,p)$ satisfies (A. 29). In addition, since $\psi = \partial\psi/\partial t = 0$ at $t = 0$ the operational form of the wave equation is (Jeffreys & Jeffreys 1946)

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2} = p^2\Psi. \tag{A. 31}$$

One therefore seeks a function Ψ satisfying equation (A. 31) together with the appropriate boundary conditions. In particular, since ψ has discontinuities of amount $\mathfrak{F}(t+x)$ along $y = 2b$, $y = 0$, $x<0$, the appropriate discontinuities in Ψ will be of amount $F(p)\,e^{px}$. Once Ψ has been found it can be verified that $\psi(t)$, obtained from (A. 29), is a solution of the wave equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial t^2} \tag{A. 32}$$

that satisfies the physical boundary conditions. It then follows at once that

$$\left.\begin{aligned} \phi &= F(p)\,e^{px}+\Psi(x,y,p) \quad (-2b<y<0) \\ &= \Psi(x,y,p) \quad (y<-2b,\, y>0), \end{aligned}\right\} \tag{A. 33}$$

represents, operationally, the whole potential field and is continuous for $x<0$ in the transition between the two regions. Moreover, ϕ clearly satisfies (A. 31) provided only that Ψ does so.

In particular, whereas the derivation of equation (A. 31) for Ψ depends on the conditions $\psi = \partial\psi/\partial t = 0$ at $t = 0$, the function $F(p)\,e^{px}$ and its interpretation $\mathfrak{F}(t+x)$ always satisfy equations (A. 31) and (A. 32) respectively, provided that the integral

$$\frac{1}{2\pi i}\int_{y-i\infty}^{y+i\infty}\frac{e^{zt}}{z}F(z)\,dz \tag{A. 34}$$

exists and is continuous for $t>0$. Further, since the differential equation is linear, any two solutions may be superposed so that $\mathfrak{F}(t+x)$ may contain any number of finite discontinuities travelling with the wave.

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